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Some New and Old (Unsolved) Problems and Conjectures on Factorization Theory, Dynamics and Functional Equations of Meromorphic Functions

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Abstract: Some old and new problems and conjectures in factorization theory, dynamics, functional equations and differential polynomial of meromorphic and entire functions are discussed. It is hoped that this can stimulate the young researchers' interests to these problems.

Key words: Nevanlinna's value distribution theory; factorization theory; dynamics; functional equation; differential polynomial

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0 Introduction

The purpose for posing these old and new problems and conjectures arising in studying the Nevanlinna's value distribution theory and its applications is to stimulate peers' interest to further study of the theory and properties of entire and meromorphic functions of one complex variable. Be reminded that one can find studies and results on similar topics for functions of several complex variables or p-adic functions of several complex variables [1-5]. For simplicity, here we shall address the problems and conjectures for (non-constant) entire or meromorphic functions of one complex variable only, and most of them were proposed by the authors and their co-workers earlier. We refer the reader to [6-8] for the Nevanlinna's theory and its associated notations. Problems and conjectures will be stated in four different topics: (i) factorization and fixed points of entire functions; (ii) dynamics of two permutable entire functions; (iii) functional equations of Diophantine type over functions field; (iv) complex differential equations. Sources, definitions, and results relevant to these problems and conjectures will be stated or cited, with references. One can find that some of open problems and conjectures were posed in [9] and as far as the present authors know of that no significant progresses or results towards those problems or conjecturrs have been obtained.

1 Factorizations and Fixed Points of Entire Functions

Definition 1 [10] Let F, f and g be non-constant entire functions. The expression or composition: $F = f(g)$ (or $f \circ g$) is called a factorization of F, with f and g the left and right factor, respectively. F is called a prime (pseudo-prime) function iff whenever F has a factorization: $F = f(g)$ implies that either f or g is a linear function (polynomial).

There have been many sufficient conditions or criteria to judge whether or not a certain function is prime or pseudo-prime, but no necessary condition or criterion for such a verification has been found yet. N. Stein-

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metz[11] proved a general sufficient condition to judge a transcendental entire function $F(z)$ is pseudo-prime.

Theorem 1 Let $F(z)$ be a transcendental meromorphic solution of the following linear differential equation

$$w^{(n)} + a_{n-1}(z) w^{(n-1)} + \cdots + a_0(z) w + a(z) = 0,$$

where $a_{n-1}(z), \cdots, a_0(z), a(z)$ are rational functions. Then $F(z)$ is pseudo-prime.

Question 1 What is a necessary condition for an entire function to be prime or pseudo-prime?

If F is a pseudo-prime entire function, $P(F)$ may not be pseudo-prime. Song Guodong et al.[12] proved the following result.

Theorem 2 If $F(z)$ is a pseudo-prime entire function, $n(\geqslant 3)$ is odd integer number, then $G(z) = F^n(z)$ is also pseudo-prime function.

Theorem 3 $F(z) = (\sin z) e^{\cos z}$ is prime, however, $F^2(z)$ is not pseudo-prime.

Obviously, $F(z) = (\sin z) e^{\cos z}$ is of infinite order. Liao Liangwen et al.[13] gave a prime function of finite order, whose even powers are not pseudo-prime. If P is a non-constant polynomial such that $P'(z)$ has only one zero z_0, then $P(z) = a(z - z_0)^n + b$. It follows Theorem 2, Theorem 3 and [13]'s result that if n is odd number and $F(z)$ is a pseudo-prime function, then $P(F)$ is still pseudo-prime; if n is even number, then exist some prime functions $F(z)$ such that $P(F)$ is not pseudo-prime.

Conjecture 1 Let F be a pseudo-prime entire function and P be a non-constant polynomial such that $P'(z)$ has two distinct zeros. Then $P(F)$ is pseudo-prime.

Conjecture 2[10] Let F be a pseudo-prime entire function and P be a non-constant polynomial. Then $F(P)$ remains to be pseudo-prime.

Note that one can exhibit some examples to show that, in general.

Definition 2 Let f, F, g, G, h, H, k and K denote entire functions. h is called a (right) factor of H if and only if $H = k(h)$ and will be noted as $h \mid H$. And K is called a greatest common (right) factor of F and G iff $K \mid F, K \mid G$ and if h is any other common factor of F and G, then $h \mid K$. If the only common factor of F and G is a linear function, then two entire functions F and G to be relatively prime. K is called a least common multiplier of F and G iff $F \mid K$ and $G \mid K$, and if H is any other function satisfying $H \mid F$ and $H \mid G$, then $K \mid H$.

It has been shown that any collection of entire functions $E = \{F_\alpha\}$, under a slightly general definition of a (right) factor, there always exists a common factor for E, which may be a linear function[14]. However, it follows from an example in [15] that a common multiplier of two functions may not exist.

Question 2 What is a necessary and sufficient condition for two entire functions F and G to be relatively prime? That is the only common factor of F and G is a linear function.

It has been known that $F(z) = z + p(z) \exp\{\alpha(z)\}$ is a prime function, where p is a polynomial $(\not\equiv 0)$ and $\alpha(z)$ is a non-constant entire function[16]. Which immediately shows that if neither f nor g is linear, then a transcendental function of the form $f(g)$ must have infinitely many fixed points. In fact, it had been a conjecture for long while that such a function $f(g)$ must have infinitely many fixed points[16-17]. As a further study of the quantitative estimate of the number of fixed points of composite functions, Yang Chungchun et al.[17] proved the following result.

Theorem 4 Let $f(z)$ and $g(z)$ be two transcendental entire functions and $P(z)$ be a non-constant polynomial. Then there exists a set I having the lower logarithmic density one such that

$$\lim_{r\to\infty, r\in I} \frac{N(r+\eta, 1/(f(g) - P))}{T(r, g)} = \infty,$$

where $\eta = 7crv(r, g)^{-\gamma}$, $1/2 < \gamma < 1$ and $c = \arctan(16/13) + \pi/2$.

The following conjecture was raised.

Conjecture 3[17-18] Let f be a transcendental meromorphic function, g be a nonlinear function, $a(z)$ be a small function of f. Then

$$N(r, 1/(f(g) - a)) \neq o(T(r, f(g))) \text{ as } r\to\infty$$

F. Gross[19] posed the following question.

Question 3 Given any entire function f, does there exist a polynomial Q such that $f + Q$ is prime?

Above question was solved by Y. Noda[20]. More precisely, he proved the following result.

Theorem 5 Let $f(z)$ be a transcendental entire function. Then the set

$$S(f) = \{a \mid a \in \mathbf{C}, f(z) + az \text{ is not prime}\}$$

is at most countable.

M. Ozawa et al. [21] proved that for a certain kind of entire functions f, that the cardinality of $S(f)$ is at most 2.

Liao Liangwen et al. [22] proved the following result.

Theorem 6 Let f be a transcendental entire function of finite order. Denote $C_f=\{f(z)\mid f'(z)=0\}$, i. e. C_f is the set of all critical values of f. If C_f is a finite set, then for any constant $a\neq 0$, $f(z)+az$ is prime, i. e., the cardinality of $S(f)$ is at most 1.

Theorem 7 If f is an entire function of finite order with C_f being a finite set. Then for any constant $a\neq 0$ and any nonconstant polynomial P, $P(f(z))+az$ is prime, the cardinality of $S(P(f))$ is at most 1.

Conjecture 4 For any transcendental entire function f, the maximal cardinal number of the exceptional set $S(f)$ is 2.

2 Dynamics of Two Permutable Entire Functions

It has been a well known result that if two polynomials p, q are permutable, then p, q have the same Julia and Fatou sets. Thus far, the same conclusion holds for two permutable transcendental entire functions f, g that are of bounded type, that is the sets of finite singularities (critical values and asymptotic values) of both f and g are bounded [23]. T. W. Ng [24], Liao Liangwen et al. [25] proved for some transcendental entire function f, if g is permutable with f, then $g=af^n+b$, where a, b are constants. Thus, f and g have the same Julia and Fatou sets. A natural question is

Question 4 What are the relations between two permutable transcendental entire functions f and g? Is there a complete classification of all pairs of nonlinear permutable entire functions?

Liao Liangwen et al. [26] proved that

Theorem 8 Let f and g be two permutable transcendental entire functions. If there exist a transcendental entire functions h, a rational function f_1 and a function g_1 that is analytic in the range of h, such that $f(z)=f_1(h(z))$ and $g(z)=g_1(h(z))$, then $F(g)\subset F(f)$, where $F(f)$ denote the Fatou set of f.

It is easy to get the following result from Theorem 8.

Corollary 1 Let f and g be two permutable pseudo-prime transcendental entire functions. If they have a common transcendental factor, then they have same Julia and Fatou sets.

Here, we mention I. N. Baker's question [27]:

Question 5 Let f and g be two transcendental entire functions. If f and g are permutable, is $F(f)=F(g)$?

3 Functional Equations of Diophantine Type

Conjecture 5 [28] If a Diophantine equation: $F(x,y)=0$, where F is an irreducible polynomial of degree higher than 3, with rational numbers as the coefficients, has none or finitely many rational solutions, then the corresponding equation $F(f,g)=0$ has none or finitely many non-constant transcendental meromorphic solutions f and g. Here the two pairs of solutions (f,g) and $(f(h),g(h))$ for any non-constant entire h are said to be equivalent.

Conjecture 6 Let P denote a non-constant polynomial. The only transcendental entire solutions for the equation of the form: $f^2(z)+p(z)g^2(z)=1$ are pairs: $(\pm f,\pm g)$, with $f=\cos(\sqrt{p(z)}u(z))$ and $g=\sin(\sqrt{p(z)}u(z))/\sqrt{p(z)}$, where $u(z)$ is an entire function.

Definition 3 Let $P(z,f,g)$ be a polynomial in f and g, with entire or meromorphic functions of z as the coefficients. A pair of meromorphic solutions (f,g) of the equation $P(z,f,g)=0$ is called admissible if all the coefficients of the equation are small functions of f and g.

See [29] for some results on the existence of admissible solutions of functional equations of the form:

$$f^n+a_1f^{n-m}+b_1=c(g^n+a_2g^{n-m}+b_2),$$

where $a_i(i=1,2)$, $b_i(i=1,2)$ and c are meromorphic functions and none of them is identically zero.

Note, in the above result, n is assumed to be greater than 8, with $m\geqslant 2$ and $n>2m+3$. It seems to be quite difficult to deal with the above functional equations when the degree of f or g is less than 8.

Conjecture 7 [29] Let p, q be two polynomials both having at least three distinct zeros, and $a(z)$ be any non-constant meromorphic function. Then the functional equation: $p(f) = a(z)q(g)$ has no admissible solutions f and g.

Conjecture 8 [30] The functional equation: $f^2 = g^3 + b(z)g + c$, where b is a non-constant meromorphic function and c is a constant, has no pair of admissible solutions.

4 Complex Differential Polynomial and Differential Equations

In complex differential equation theory, an interesting research problem is to investigate whether an algebraic differential equation can be reduced to some standard forms if the algebraic differential equation admits a transcendental meromorphic solution. This kind of results is usually called as a Malmquist-Yosida type theorem. J. Malmquist [31], K. Yosida [32], I. Laine [33], Yang chungchun [34], E. Hille [35] and N. Steinmetz [36] proved some Malmquist-Yosida type theorems of the first order algebraic differential equations. Liao Liangwen et al. [37] got a Malmquist-Yosida type theorem of a certain type of the second order algebraic differential equations. It is quite difficult to find a Malmquist-Yosida type theorem for an arbitrary second order algebraic differential equation. For the second-order differential equation

$$f'' = R(z, f, f'), \tag{1}$$

where R is rational in z, f and f', a classical and yet unsolved conjecture is

Conjecture 9 [38] If the equation (1) has a transcendental meromorphic solution, then the equation can be transformed to the form:

$$f'' = N(z, f)(f')^2 + M(z, f)f' + L(z, f),$$

where L, M, N are rational functions in their arguments.

It is useful but difficult to study the properties of the meromorphic or entire solutions of some differential equations. Some questions are related to other fields. The following conjecture is related to Brück's conjecture in the uniqueness theory of meromorphic functions.

Conjecture 10 If f is an entire solution of the following differential equation

$$f^{(k)} - e^{g(z)}f - 1 = 0,$$

where $g(z)$ is a transcendental entire function, then the super order $\sigma_2(f)$ of f is infinite. The definition of $\sigma_2(f)$ is following

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log \log T(r, f)}{\log r}.$$

W. K. Hayman [39] began studying the value distribution of the differential polynomials of meromorphic functions by proving that if f is a transcendental entire function, then $f'f^n$ assumes every non-zero complex number infinitely many times, provided that $n \geqslant 2$. Since then, there are many research publications [40-44] regarding the value distributions of the differential polynomials of meromorphic functions. Recently, Liao liangwen et al. [45] got a general result about the value distributions of the differential polynomials of meromorphic functions.

Theorem 9 Let f be a transcendental entire function, $p_n(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0$ is a polynomial with degree n, $q_m(f) = b_m f^m + b_{m-1} f^{m-1} + \cdots + b_0$ is a polynomial with degree m and $n \geqslant m + 1$. Then $f'p_n(f) + q_m(f)$ assumes every complex number α infinitely many times, except a possible value $q_m(-a_{n-1}/(na_n))$. On the other hand, if $f'p_n(f) + q_m(f)$ assumes the complex value $q_m(-a_{n-1}/(na_n))$ finitely many times, then either

(i) $p_n(z) = a_n(z + a_{n-1}/(na_n))^n$, $q_m(z)$ is a constant polynomial, which is $q_m(-a_{n-1}/(na_n))$, and $f + a_{n-1}/(na_n)$, f' have only finitely many zeros; or

(ii) $f(z) = Ae^{Bz} + a_{n-1}/(na_n)$, where A, B are some constants; only when q_m is non-constant and f is of finite order.

Question 6 Is Theorem 9 valid for a meromorphic functions?

Question 7 If f' is replaced by $f^{(k)}$ $(k \geqslant 2)$ in Theorem 9, is the conclusion in Theorem 9 still true?

Finally, the reader is suggested to read the following related monographs [9-10, 46-49] for resolving the posed conjectures or problems, as well as finding some new problems or conjectures for further studies.

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亚纯函数分解理论,动力系统和函数方程中的一些新的和未解决的老问题与猜想

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摘要: 提出了一些亚纯函数分解理论、动力系统、函数方程和微分多项式老的和新的问题与猜想,希望能够激起年轻研究人员对这些问题的兴趣.

关键词: Nevanlinna 值分布理论; 函数分解理论; 动力系统; 函数方程; 微分多项式

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