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The Non-Existence of Transcendental Meromorphic Solutions of Some Certain Type of Second Order Algebraic Differential Equations

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Abstract: By using normal family theory of meromorphic function, the problem on non-existence of meromorphic solutions of some types of second-order algebraic differential equations are investigated. Some restriction conditions which can ensure the equation admits no transcendental meromorphic solutions are obtained.

Key words: algebraic differential equations; meromorphic solutions; growth order

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0 Introduction and Results

Let $f(z)$ be a meromorphic function. We assume that the reader is familiar with Nevanlinna's value distribution theory of meromorphic functions[1-3]. We denote the order of $f(z)$ by $\rho(f)$. $S(r,f)$ denotes any quantity that satisfies the condition: $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside an exceptional set of r of finite linear measure.

In this paper, we denote the spherical derivative of meromorphic function $f(z)$ by $f^{\#}(z)$, where

$$f^{\#}(z) := |f'(z)|/(1 + |f(z)|^2),$$

and define

$$A(r,f) = \frac{1}{\pi}\iint_{|z|\le r} [f^{\#}(z)]^2 \mathrm{d}x\mathrm{d}y.$$

A differential polynomial in f is usually expressed as

$$P(z,f,f',\cdots,f^{(n)}) = \sum_{\lambda\in I} M_\lambda(z,f) = \sum_{\lambda\in I}\alpha_\lambda(z) f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n},$$

where I is a finite set of multi-indices $(\lambda_0,\lambda_1,\cdots,\lambda_n) = \lambda$ and $\alpha_\lambda(z)$ $(\lambda\in I)$ are meromorphic functions.

The degree γ_{M_λ} and the weight Γ_{M_λ} of M_λ are defined by

$$\gamma_{M_\lambda} := \lambda_0 + \lambda_1 + \cdots + \lambda_n,$$
$$\Gamma_{M_\lambda} := \lambda_0 + 2\lambda_1 + \cdots + (n+1)\lambda_n.$$

The degree γ_P and the weight Γ_P of P are defined by

$$\gamma_P = \max_{\lambda\in I}\gamma_{M_\lambda}, \Gamma_P = \max_{\lambda\in I}\Gamma_{M_\lambda}.$$

If $\gamma_{M_\lambda} = \gamma_P$, then we say that the term $M_\lambda(z,f)$ is a dominant term of $P(z,f,f',\cdots,f^{(n)})$. A meromorphic solution $f(z)$ of following algebraic differential equation

$$P(z,f,f',\cdots,f^{(n)}) = 0 \quad (1)$$

is called admissible, if $T(r,\alpha_\lambda(z)) = S(r,f)$ hold for all coefficients $\alpha_\lambda(z)$ of equation (1).

It is very interesting and difficult to investigate the meromorphic or entire solution of algebraic differential equations[4-13]. In particular, some papers obtained some non-existence results of certain types of algebraic differential equations. In 1980, F. Gackstatter et al.[4] considered the following specific algebraic differential equation

$$(f')^n = Q(z,f),$$

where $Q(z,f)$ is a polynomial in f with meromorphic co-

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efficients, and conjectured that the equation has no admissible meromorphic solution when

$$\deg_f Q(z,f) \leqslant n-1.$$

In 1990, He Yuzan et al.[5] gave a positive answer to the above conjecture. In 1991, K. Ishizaki[6] proved a more general result as follows.

Theorem A The differential equation

$$P(z,f') = Q(z,f),$$

where $P(z,f')$, resp. $Q(z,f)$, is a polynomial in f', resp. in f, with meromorphic coefficients and $1 \leqslant \deg_f Q(z,f) \leqslant \deg_{f'} P(z,f') - 1$, admits no admissible meromorphic solutions.

For the non-existence of meromorphic solution of the general algebraic differential equation, in 1955, H. Wittich[7] gave a classic result as follows.

Theorem B If the algebraic differential equation

$$P(z,f,f',\cdots,f^{(n)}) = 0,$$

where $P(z,f,f',\cdots,f^{(n)}) = 0$ is a differential polynomial in f with polynomial coefficients, has only one dominant term, then the equation has no transcendental entire solutions.

In 2013, Zhang Jianjuan et al.[8] obtained the following result.

Theorem C If the algebraic differential equation

$$P(z,f,f',\cdots,f^{(n)}) = 0,$$

where $P(z,f,f',\cdots,f^{(n)}) = 0$ is a differential polynomial in f with meromorphic coefficients, has only one dominant term, then the equation has no admissible transcendental meromorphic solutions satisfying

$$N(r,f) = S(r,f).$$

As one knows, research on Malmquist-Yosida type problem of algebraic differential equation is an important topic. However, for genernal second-order algebraic differential equation, Malmquist-Yosida type theorem remains open. For the following specific second-order algebraic differential equation

$$f'' = R(z,f,f'), \tag{2}$$

where R is a rational function in z, f and f', we have the following unsolved conjecture.

Conjecture 1[3] If equation (2) has a transcendental meromorphic solution, then the equation can be reduced into the form

$$f'' = L_2(z,f)(f')^2 + L_1(z,f)f' + L_0(z,f),$$

where $L_i(z,f)$ $(i=0,1,2)$ are rational functions in their variables.

Next we will consider the non-existence of transcendental meromorphic solution of equation (2). In order to state our results, we will give following definitions and notations.

Let $P(z,u,v)$ and $Q(z,u,v)$ be two irreducible polynomial functions in their variables. $\deg_z P$ denotes the degree of $P(z,u,v)$, and

$$\deg_z R = \deg_z \frac{P(z,u,v)}{Q(z,u,v)} := \max\{\deg_z P, \deg_z Q\}$$

denotes the degree of $P(z,u,v)/Q(z,u,v)$ in z.

For convenience, in (2), we assume

$$R(z,f,f') = [L_n(z,f)(f')^n + \cdots + L_1(z,f)f' + L_0(z,f)] / [M_m(z,f)(f')^m + \cdots + M_1(z,f)f' + M_0(z,f)],$$

where $L_i(z,f)$ $(i=0,1,\cdots,n)$, $M_j(z,f)$ $(j=0,1,\cdots,m)$ are polynomial functions in z and f.

When $n > m+2$, we have the following result.

Theorem 1 If $n > m+2$, then the algebraic differential equation (2) admits no transcendental meromorphic solution f of order $\rho(f) > 2 + 2\deg_z R$.

Theorem 2 If $n > m+2$, then the algebraic differential equation (2) admits no transcendental entire solution f of order $\rho(f) > 1 + \deg_z R$.

Now we give the following open problem.

Open problem If $n \leqslant m+2$, then the algebraic differential equation (2) admits no transcendental meromorphic solution f of order $\rho(f) > 2 + 2\deg_z R$.

1 Some Lemmas

In order to prove our results, next lemmas will be needed.

Lemma 1 Let $f(z)$ be a meromorphic function and $\rho := \rho(f) > 2$. Then for any $\varepsilon > 0$ and $0 < \lambda < ((\rho-2)/2)\varepsilon$, there exist points $z_k \to \infty$ $(k \to \infty)$, such that

$$\lim_{k\to\infty} (f^{\#}(z_k))^{\varepsilon} / |z_k|^{\lambda} = +\infty$$

Proof Suppose that the conclusion is invalid, then there exist $M > 0$, such that for any $z \in \mathbf{C}$,

$$(f^{\#}(z))^{\varepsilon} \leqslant M|z|^{\lambda}. \tag{3}$$

By (3) we can get

$$A(t,f) = \frac{1}{\pi}\iint_{|z|\leqslant t} [f^{\#}(z)]^2 \mathrm{d}x\mathrm{d}y \leqslant \frac{M^{2/\varepsilon}}{\pi}\iint_{|z|\leqslant t} |z|^{2\lambda/\varepsilon}\mathrm{d}x\mathrm{d}y = O(|t|^{2+2\lambda/\varepsilon}).$$

Thus we have the following estimation for characteristic function

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t}\mathrm{d}t \leqslant O(r^{2+2\lambda/\varepsilon}) .$$

Therefore, the order $\rho(f)$ can be estimated as $\rho \leqslant 2 + 2\lambda/\varepsilon$, so $\lambda \geqslant ((\rho-2)/2)\varepsilon$. This is a contradiction.

Lemma 2 [14] Let $f(z)$ be a holomorphic function and $\sigma > -1$. If $f^{\#}(z) = O(r^{\sigma})$, then we have

$$T(r,f) = O(r^{\sigma+1}) .$$

By Lemma 2, we can prove following result easily.

Lemma 3 Let $f(z)$ be a holomorphic function and $\rho := \rho(f) > 1$. Then for any $\varepsilon > 0$ and $0 < \lambda < (\rho-1)\varepsilon$, there exist points $z_k \to \infty (k \to \infty)$, such that

$$\lim_{k\to\infty} (f^{\#}(z_k))^{\varepsilon} / |z_k|^{\lambda} = +\infty$$

Proof Suppose that the conclusion is invalid, then there exist $M > 0$, such that for any $z \in \mathbf{C}$, we have

$$(f^{\#}(z))^{\varepsilon} \leqslant M|z|^{\lambda} .$$

Namely

$$f^{\#}(z) = O(r^{\lambda/\varepsilon}) .$$

By Lemma 2, we have

$$T(r,f) = O(r^{\lambda/\varepsilon+1}) .$$

Therefore, $\rho(f)$ can be estimated as $\rho \leqslant 1 + \lambda/\varepsilon$. This is a contradiction.

Lemma 4 [15] Let F be a family of meromorphic functions on the unit disc, α is a real number. Then F is not normal on the unit disc if and only if there exist, for each $-1 < \alpha < 1$,

(i) a number r, $0 < r < 1$;

(ii) a sequence points $\{w_k\}$, $|w_k| < r$;

(iii) a sequence $\{f_k\}_{k\in \mathrm{N}} \subset F$;

(iv) a positive sequence $\{\rho_k\}$, $\rho_k \to 0$,

such that $g_k(\zeta) := \rho_k^{\alpha} f_k(w_k + \rho_k\zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$. In particular, we may choose w_k and ρ_k properly such that

$$\rho_k \leqslant \frac{2}{[f_k^{\#}(w_k)]^{1/(1+|\alpha|)}}, f_k^{\#}(w_k) \geqslant f_k^{\#}(0) .$$

2 Proof of Theorems

Proof of Theorem 1 Suppose that f is a transcendental meromorphic solution of order $\rho(f) > 2 + 2\deg_z R$ of equation (2). We first prove that $m = 0$. In fact, if the assertion is invalid, then we assume that $m \geqslant 1$.

Since $\rho(f) > 2 + 2\deg_z R$, by Lemma 1 we know that for $\varepsilon = 1$ and $0 < \lambda < (\rho-2)/2$, there exist points $z_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k\to\infty} (f^{\#}(z_k))^{\varepsilon} / |z_k|^{\lambda} = +\infty \tag{4}$$

Thus $f^{\#}(z_k) \to \infty$, as $k \to \infty$ It implies that the family $\{f(z_k + z)\}_{k\in\mathrm{N}}$ is not normal at $z = 0$. By Lemma 4, there exist a sequence β_k and a positive sequence ρ_k such that

$$|z_k - \beta_k| < 1, \rho_k \to 0 , \tag{5}$$

and $g_k(\zeta) := f(\beta_k + \rho_k\zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$. In particular, we may choose β_k and ρ_k, such that

$$\rho_k \leqslant 2/f^{\#}(\beta_k), f^{\#}(\beta_k) \geqslant f^{\#}(z_k) . \tag{6}$$

According to (4), (5) and (6), we can get the following conclusion.

For any constant $0 \leqslant \lambda < (\rho-2)/2$, we have

$$\lim_{k\to\infty} \beta_k^{\lambda}\rho_k = 0 . \tag{7}$$

By replacing z by $\beta_k + \rho_k\zeta$ in (2), we have

$$\frac{g''_k(\zeta)}{\rho_k^2} = [L_n(\beta_k+\rho_k\zeta, g_k(\zeta))\frac{g'_k(\zeta)^n}{\rho_k^n} + \cdots + L_1(\beta_k+\rho_k\zeta, g_k(\zeta))\frac{g'_k(\zeta)}{\rho_k} + L_0(\beta_k+\rho_k\zeta, g_k(\zeta))] / [M_m(\beta_k+\rho_k\zeta, g_k(\zeta))\frac{g'_k(\zeta)^m}{\rho_k^m} + \cdots + M_1(\beta_k+\rho_k\zeta, g_k(\zeta))\frac{g'_k(\zeta)}{\rho_k} + M_0(\beta_k+\rho_k\zeta, g_k(\zeta))] .$$

Because $0 \leqslant \deg_z L_i, \deg_z M_j \leqslant \deg_z R < (\rho(f)-2)/2 (0 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$, we rewrite it as the form:

$$g'_k(\zeta)^n = [M_m(\beta_k + \rho_k\zeta, g_k(\zeta))/L_n(\beta_k + \rho_k\zeta, g_k(\zeta))]\rho_k^{n-m-2}g''_k(\zeta)g'_k(\zeta)^m + o(\rho_k^{n-m-2}) .$$

Noting $n > m+2$, we can conclude from this and (7), by letting $k \to \infty$, $g'(\zeta) \equiv 0$. This is a contradiction. Thus m must be zero. Hence the equation (2) is reduced into

$$f'' = \frac{L_n(z,f)}{M_0(z,f)}(f')^n + \frac{L_{n-1}(z,f)}{M_0(z,f)}(f')^{n-1} + \cdots + \frac{L_0(z,f)}{M_0(z,f)} . \tag{8}$$

If $n > 2$, then by similar argument as above, replacing z by $\beta_k + \rho_k\zeta$ in (8), we obtain

$$g'_k(\zeta)^n = \frac{M_0(\beta_k+\rho_k\zeta, g(\zeta))}{L_n(\beta_k+\rho_k\zeta, g_k(\zeta))}\rho_k^{n-2}g''_k(\zeta) + o(\rho_k^{n-2}) .$$

From this and (7), by letting $k \to \infty$, we get $g'(\zeta) \equiv 0$. Thus we get a contradiction. The proof of Theorem 1 is completed.

Proof of Theorem 2 By Lemma 3 and the similar discussion in Theorem 1, we can easily prove theorem 2.

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某类 2 阶代数微分方程超越亚纯解的非存在性

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摘要: 运用亚纯函数正规族理论,对一类 2 阶代数微分方程亚纯解的存在性问题进行了研究,得到了确保方程不存在超越亚纯解时的一些限制条件.

关键词: 代数微分方程; 亚纯解; 增长级

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