文章编号:1000-5862(2018)01-0001-011

The Recent Progress in Nevanlinna Theory

RU Min

(Department of Mathematics,University of Houston,Houston TX 77204,USA)

Abstract: Being based on Picard´s and Borel´s theorems,R. Nevanlinna published his paper and evolved a theory affiliated with his name. Since then the Nevanlinna theory has become an important subject in complex analysis,complex geometry and several complex variables. Some important developments in the past research are recalled in this paper,as well as a partial survey on some most recent progress in the study of Nevanlinna theory is given.

Key words: Nevanlinna theory; the First Main Theorem(FMT); the Second Main Theorem(SMT)

中图分类号: O 174.5 **文献标志码:** A **DOI:** 10.16357/j.cnki.issn1000-5862.2018.01.01

0 Introduction

About ninety years ago,Rolf Nevanlinna [1] extended the classical theorems of Picard and Borel,and developed the value distribution theory of meromorphic functions,which is now called Nevanlinna theory. In many ways,Nevanlinna theory is a best possible theory for both meromorphic and entire functions,and it has been used to prove numerous important results about meromorphic and entire functions. Nevanlinna developed his theory in a series of papers from 1922—1925, and literature [1] is considered his most important paper. In 1943,H. Weyl [2] made the following comment about literature [1]: "The appearance of this paper has been one of the few great mathematical events in our century".

The core of Nevanlinna theory consists of two Main Theorems: the First Main Theorem (FMT) and the Second Main Theorem (SMT). The First Main Theorem is considered to be a non-compact version of Poincaré duality,and we now have a satisfactory theory for it. So this paper mainly devote to establishing the Second Main-type Theorems.

1 Nevanlinna´s Second Main Theorem and Chern´s Geometric Extension

The Fundamental Theorem of Algebra states that for every non-constant complex polynomial P, $\deg P = n_P(a)$,where $\deg P$ is the degree of P which measures the growth of P and $n_P(a)$ is the number of the roots of $P(z) = a$ on the complex plane $\mathbf{C}$,counting multiplicities. It is known that entire functions,or more generally the meromorphic functions on $\mathbf{C}$,behave in many ways similar to the polynomials. To extend the Fundamental Theorem of Algebra,the firrst step is to find the measurement of the growth of f. Hadamard made the first discovery in this direction. Similar to the algebraic case,given an entire function,there are two different ways of measuring its rate of growth—its maximum modulus on the disc of radius r(viewed as a function of r) and the maximum number of times at the value in the image is taken on this disc. The insight is that these two rates of growth are essentially the same,the former being roughly the exponential of the latter. R. Nevanlinna [1],in 1929, found the right substitute for the maximum modulus. He introduced the characteristic function $T_f(r)$ to measure the growth of the meromorphic function f. Starting from the Poisson-Jensen formula,he was able to derive a more subtle growth estimate for meromorphic functions in what he called the Second Main Theorem. It gives a quantitative version of the classical Picard´s theorem for meromorphic functions.

收稿日期:2017-07-28

基金项目:Simons 数学基金(521604)和美国国家安全局数学课题(H98230-11-1-0201)资助项目.

作者简介:汝 敏(1963-),男,江苏吴江人,美国休斯顿大学终身教授,主要从事复分析、丢番图逼近和微分几何方面的研究. E-mail: minru@math.uh.edu

We now describe his theory. Let f be a meromorphic function on $\mathbf{C}$. Denote the number of poles of f on the disc $\{z \mid |z| < r\}$ by $n_f(r,\infty)$, counting multiplicity. We then define the counting function $N_f(r,\infty)$ to be

$$N_f(r,\infty) = n_f(0,\infty)\log r + \int_0^r [n_f(r,\infty) - n_f(0,\infty)]\frac{\mathrm{d}t}{t},$$

here $n_f(0,\infty)$ is the multiplicity if f has a pole at $z = 0$. For each complex number a, we define the counting function $N_f(r,a)$ to be

$$N_f(r,a) = N_{1/(f-a)}(r,\infty).$$

The Nevanlinna's proximity function $m_f(r,\infty)$ is defined by

$$m_f(r,\infty) = \int_0^{2\pi}\log^+|f(re^{i\theta})|\frac{\mathrm{d}\theta}{2\pi},$$

where $\log^+ x = \max\{0,\log x\}$. For any complex number a, the proximity function $m_f(r,a)$ of f with respect to a is then defined by

$$m_f(r,a) = m_{1/(f-a)}(r,\infty).$$

We note that $m_f(r,a)$ measures how close f is, on average, to a on the circle of radius r. Finally, the Nevanlinna's characteristic function (or height function) of f is defined by

$$T_f(r) = m_f(r,\infty) + N_f(r,\infty).$$

$T_f(r)$ measures the growth of f. For example: $T_f(r) = O(1)$ if and only if f is constant; $T_f(r) = O(\log r)$ if and only if f is a rational function.

The characteristic function T, the proximity function m and the counting function N are the three main Nevanlinna functions. Nevanlinna theory can be described as the study of how the growth of these three functions is interrelated. The First Main Theorem is a reformulation of the classical Poisson-Jensen formula in complex analysis.

Theorem 1(First Main Theorem) Let f be non-constant meromorphic on $\mathbf{C}$. Then, for all $a \in \mathbf{C}$,

$$T_f(r) = m_f(r,a) + N_f(r,a) + O(1),$$

where $O(1)$ is a bounded term which is independent of f.

Theorem 2(Nevanlinna's Second Main Theorem) Let $a_1,\cdots,a_q$ be a set of distinct complex numbers. Let f be a non-constant meromorphic function on $\mathbf{C}$. Then, for any for $\delta > 0$, the inequality

$$(q-1)T_f(r) + N_{\mathrm{ram},f}(r) \leqslant \sum_{j=1}^{q} N_f(r,a_j) + N_f(r,\infty) + O(\log T_f(r)) + \delta\log r \parallel_\delta,$$

where $\parallel_\delta$ means the inequality holds for all $r \geqslant r_0$ outside a set $E \subset (0,+\infty)$ (which depends on δ) with finite Lebesgue measure, and $N_{\mathrm{ram},f}(r) = N_{f'}(r,0) + 2N_f(r,\infty) - N_{f'}(r,\infty)$.

The proof of Nevanlinna's Second Main Theorem is based on the following "Logarithmic Derivative Lemma (LDL)."

Theorem 3(Logarithmic Derivative Lemma (LDL)) Let $f(z)$ be a meromorphic function. Then, for $\delta > 0$,

$$\int_0^{2\pi}\log^+\left|\frac{f'}{f}(re^{i\theta})\right|\frac{\mathrm{d}\theta}{2\pi} \leqslant \left(1 + \frac{(1+\delta)^2}{2}\right)\log T_f(r) + \frac{\delta}{2}\log r + O(1) \parallel_\delta.$$

In 1960, Shiing-Shen Chern[8] extended Nevanlinna's SMT to holomorphic mappings $f: \mathbf{C} \to M$ where M is a compact Riemann surface. Note that every meromorphic function f on $\mathbf{C}$ can be viewed as a holomorphic map $f: \mathbf{C} \to \mathbf{P}^1$. If we use the chordal distance on $\mathbf{P}^1$, then the proximity function can be re-formulated as, for any $a \in \mathbf{P}^1$,

$$m_f(r,a) = \int_0^{2\pi}\log\frac{1}{\|f(re^{i\theta}),a\|}\frac{\mathrm{d}\theta}{2\pi}.$$

Thus, to extend the theory from $\mathbf{P}^1$ to M, the first thing is to find a "suitable" distance function on M such that the "First Main Theorem" holds. Let $\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$, $\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$, $\partial u = \frac{\partial u}{\partial z}\mathrm{d}z$, $\bar{\partial} u = \frac{\partial u}{\partial \bar{z}}\mathrm{d}\bar{z}$, $\mathrm{d} = \partial + \bar{\partial}$, and $\mathrm{d}^c = \sqrt{-1}(\bar{\partial} - \partial)/(4\pi)$. Note that $\mathrm{dd}^c = \sqrt{-1}\partial\bar{\partial}/(2\pi)$. Chern proved the existence of the distance by solving the Poisson's equation on M

$$-2\mathrm{dd}^c u = \omega / c \tag{1}$$

for any given positive $(1,1)$-form ω on M, where $c = \int_M \omega$. The result of Chern states that the equation has a solution with logarithmic singularity at a (any) given point on M. We denote the solution as $u(x,a)$ when $a \in M$ is given. Using this distance function, we define, for $f: \mathbf{C} \to M$ and $a \in M$,

$$m_f(r,a) = -\int_0^{2\pi} u(f(re^{i\theta}),a)\frac{\mathrm{d}\theta}{2\pi}.$$

If we let, for a given positive $(1,1)$-form $\omega = a(z)\dfrac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}$ on M,

$$T_{f,\omega}(r) = \int_0^r \frac{dt}{t}\int_{|\zeta|\le t} f^*\omega,$$

then (1) gives, by using the Green-Jensen formula[4], the following First Main Theorem, for any $a \in M$,

$$T_{f,\omega}(r) = m_f(r,a) + N_f(r,a) + O(1).$$

Note that the function $u(x,a)$ on M can be obtained by using the modern algebraic geometry language: Since the divisor $D = (a)$ on M is ample, there is a metric (norm $\|\cdot\|$) on $\mathscr{O}_M(D)$ where $\mathscr{O}_M(D)$ is the line bundle associated to D. Let s be the canonical section of $\mathscr{O}_M(D)$ (i.e. $(s) = D$). Then we can take

$$u(x,a) := \log\|s(x)\|.$$

Indeed, from the definition of the first Chern-form,

$$-2dd^c u = -dd^c\log\|s(x)\|^2 = c_1(\mathscr{O}_M(D)),$$

where $c_1(\mathscr{O}_M(D))$ is the first Chern form of $\mathscr{O}_M(D)$ with respect to the given metric. Hence the Poisson's equation (1) is satisfied for $\omega := c_1(\mathscr{O}_M(D))$. Furthermore, by the Poincare-Lelong formula[4], we have, in terms of the currents,

$$-dd^c\log[\|s(x)\|^2] + D = c_1(\mathscr{O}_M(D)).$$

Therefore, from the Green-Jensen's formula, we immediately get the FMT, for any $a \in M$,

$$T_{f,\omega}(r) = m_f(r,a) + N_f(r,a) + O(1)$$

with $\omega := c_1(\mathscr{O}_M(D))$ and $D = (a)$. Note that, if ω_1, ω_2 are two positive $(1,1)$ forms, then ω_1/ω_2 is bounded since M is compact, so the growth of $T_{f,\omega_1}(r)$ and the growth of $T_{f,\omega_2}(r)$ are the same.

Theorem 4[3] (Chern's SMT) Let M be a compact Riemann surface. Let $\omega = h\dfrac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}$ be a positive $(1,1)$ form on M. Let $f: \mathbf{C} \to M$ be a non-constant holomorphic map. Let $a_1, \cdots, a_q$ be distinct points on M. Then, for every $\delta > 0$, $\sum_{j=1}^q m_f(r, a_j) + T_{f,\mathrm{Ric}(\omega)}(r) + N_{f,\mathrm{ram}}(r) \le O(\log T_{f,\omega}(r)) + \delta\log r \,\|_\delta$, where $\mathrm{Ric}(\omega) = dd^c\log h$.

We discuss the consequences of the Theorem 4. By the uniformization theorem, a (simply connected) compact Riemann surface M is either biholomorphic to the Riemann sphere $\mathbf{P}^1$, the torus or the surface of genus ≥ 2.

When $M = \mathbf{P}^1$, the Fubini-Study form ω on $\mathbf{P}^1$ is given in terms of an affine coordinate ω by

$$\omega = \frac{1}{(1+|\omega|^2)^2}\frac{\sqrt{-1}}{2\pi}d\omega \wedge d\bar{\omega} = dd^c\log(1+|\omega|^2).$$

Thus $\mathrm{Ric}(\omega) = -2\omega$. So, for any meromorphic function f on $\mathbf{C}$ (also being regarded as a holomorphic map $f: \mathbf{C} \to \mathbf{P}^1$),

$$T_{f,\mathrm{Ric}(\omega)}(r) = T_{f,-2\omega}(r) = -2T_{f,\omega}(r),$$

where

$$T_{f,\omega}(r) = \int_0^r \frac{dt}{t}\int_{|\zeta|\le t} f^*\omega = \int_0^r \frac{dt}{t}\int_{|\zeta|\le t} \frac{|f'|^2}{(1+|f|^2)^2}\frac{\sqrt{-1}}{2\pi}d\zeta \wedge d\bar{\zeta}.$$

The characteristic function $T_{f,\omega}(r)$ above is called the Ahlfors-Shimizu characteristic function. $T_{f,\omega}(r)$ differs from the Nevanlinna's characteristic function defined earlier only by a constant. Hence Theorem 4 recovers Nevanlinna's SMT.

For the torus (elliptic) case, the canonical metric is a flat metric, i.e. there exists a positive $(1,1)$ form ω such that $\mathrm{Ric}(\omega) = 0$. So in this case, Theorem 4 implies that

$$\sum_{j=1}^q m_f(r,a_j) + N_{f,\mathrm{ram}}(r) \le \varepsilon T_{f,\omega}(r) + \delta\log r \,\|_\delta.$$

In particular, if a holomorphic map from $\mathbf{C}$ into the complex torus omits one point on the torus, then f must be constant.

Finally, for the surface of genus ≥ 2, there exists a positive $(1,1)$ form ω such that $\mathrm{Ric}(\omega)$ is also a positive $(1,1)$ form, so that $T_{f,\mathrm{Ric}(\omega)}(r) \ge 0$. Thus we have

$$T_{f,\mathrm{Ric}(\omega)}(r) \le \varepsilon T_{f,\omega}(r) + \delta\log r = \varepsilon' T_{f,\mathrm{Ric}(\omega)}(r) + \delta\log r \,\|_\delta.$$

This implies that $T_{f,\mathrm{Ric}(\omega)}(r)$ is bounded, hence f is constant. So there is no non-constant holomorphic map from $\mathbf{C}$ into M if its genus ≥ 2.

We now outline a proof of Theorem 4 here. Let $D_j = (a_j)$, $1 \le j \le q$, be the divisors corresponding to the points a_j, and let s_j be the canonical section of $\mathscr{O}_M(D_i)$ (so $(s_j) = D_j$). Motivated the Poincare metric on the punctured disc, we consider

$$\Psi = \frac{\omega}{\prod_{j=1}^q (\|s_j\|^2(\log\|s_j\|^2)^2)}.$$

Write

$$f^* \Psi = \Gamma \frac{\sqrt{-1}}{2\pi} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

Then, by the Poincaré-Lelong formula, in terms of the currents,

$$\mathrm{dd}^c [\log \Gamma] = \sum_{j=1}^{q} -\mathrm{dd}^c [\log \| f^* s_j \|^2] +$$

$$f^* \mathrm{Ric}(\omega) + D_{f,\mathrm{ram}} - \sum_{j=1}^{q} \mathrm{dd}^c [\log(\log \| f^* s_j \|^2)^2].$$

Applying the integral operator $\int_0^r \frac{\mathrm{d}t}{t} \int_{|\zeta| \leqslant t} \cdot$ to the identity above and applying the Green-Jensen's formula, we get

$$\frac{1}{2}\int_{|\zeta|=r} (\log \Gamma) \mathrm{d}\theta = \sum_{j=1}^{q} m_f(r, a_j) + T_{f,\mathrm{Ric}(\omega)}(r) +$$

$$N_{f,\mathrm{ram}}(r) - \sum_{j=1}^{q} \int_0^r \frac{\mathrm{d}t}{t} \int_{|\zeta| \leqslant t} \mathrm{dd}^c [\log(\log \| f^* s_j \|^2)^2].$$

We can normalize the metric on the line bundle so that $\| s_j \|$ are small enough that

$$\log(\log \| f^* s_j \|^2)^2 = 2\log\log(1/ \| f^* s_j \|^2).$$

Thus

$$\int_0^r \frac{\mathrm{d}t}{t} \int_{|\zeta| \leqslant t} \mathrm{dd}^c [\log(\log \| f^* s_j \|^2)^2] =$$

$$\int_{|\zeta|=r} \log\log\left(\frac{1}{\| f^* s_j \|^2}\right) \frac{\mathrm{d}\theta}{2\pi} \leqslant$$

$$\log \int_{|\zeta|=r} \log \frac{1}{\| f^* s_j \|^2} \frac{\mathrm{d}\theta}{2\pi} + O(1) =$$

$$\log m_f(r, a_j) + O(1) \leqslant \log T_{f,\omega}(r) + O(1).$$

Using the calculus lemma argument [4], we have

$$\frac{1}{2}\int_{|\zeta|=r} (\log \Gamma) \mathrm{d}\theta \leqslant O(\log T_\Gamma(r)) + \delta \log r \parallel_\delta .$$

So our goal is to estimate

$$T_\Gamma(r) = \int_0^r \frac{\mathrm{d}t}{t} \int_{|\zeta| \leqslant t} \Gamma \frac{\sqrt{-1}}{2\pi} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} =$$

$$\int_0^r \frac{\mathrm{d}t}{t} \int_{|\zeta| \leqslant t} f^* \Psi.$$

Note that if $g(M) \geqslant 2$ (i.e. $\{a_1, \cdots, a_q\}$ is an empty set), then $T_\Gamma(r) = T_{f,\omega}(r)$, hence in this case the estimate is already done. In the general case, we follow the approach by Chern-Ahlfors: by a change of variable formula,

$$\int_M n_f(r, a) \Psi(a) = \int_{|z|<r} f^* \Psi.$$

So, using the First Main Theorem,

$$\int_0^r \frac{\mathrm{d}t}{t} \int_{|z|<t} f^* \Psi = \int_M N_f(r, a) \Psi(a) \leqslant$$

$$\int_M T_{f,\omega}(r) \Psi(a) + O(1) = c T_{f,\omega}(r) + O(1),$$

where $c = \int_M \Psi$ is a constant. Hence $T_\Gamma(r) \leqslant c T_{f,\omega}(r) + O(1)$. This finishes the proof.

Note that there is an alternative method of estimating $T_\Gamma(r)$ in terms of $T_{f,\omega}(r)$. It is based on the calculation of (negative) curvature

$$\mathrm{dd}^c \log\left(\frac{1}{\log \| s_j \|^2}\right)^2 \geqslant 2\left\{\frac{c\omega}{\| s_j \|^2 (\log \| s_j \|^2)^2} - \varepsilon\omega\right\} \tag{2}$$

for some positive constant c. This important alternative method allows Griffiths and his school to make the great progress in 1970's in extending Chern's result. This leads the discussion in the next section.

2 The Results of Carlson and Griffiths

In 1972, J. Carlson and P. Griffiths [5] extended Chern's result to differentially non-degenerate holomorphic mapping $f: \mathbf{C}^n \to M$ (i.e. the Jacobian $J_f(z) \not\equiv 0$), where M is an algebraic projective variety and $n \geqslant \dim M$ (we can just assume that $n = \dim M$).

We introduce some notation. We first recall the definition of height, proximity and counting functions. Recall the normalized Euclidean form on $\mathbf{C}^n$ is $\phi_0 = \mathrm{dd}^c |z|^2$. Denote by $\omega_0 = \mathrm{dd}^c \log |z|^2$, and the Poincare-form $\sigma = (\mathrm{d}^c \log |z|^2) \wedge (\mathrm{dd}^c \log |z|^2)^{n-1}$. Then $\int_{S_r} \sigma = 1$, where S_r is the ball of radius r. Let (L, h) be an Hermitian line bundle on M. Define

$$T_{f,L}(r) = \int_0^r \frac{\mathrm{d}t}{t} \int_{B_t} f^* c_1(L, h) \wedge \omega_0^{n-1} =$$

$$\int_0^r \frac{\mathrm{d}t}{t^{2n-1}} \int_{B_t} f^* c_1(L, h) \wedge \phi_0^{n-1}, \tag{3}$$

$$m_f(r, D) = \int_{S_r} \log \frac{1}{\| f^* s \|} \sigma, \tag{4}$$

where $s \in H^0(M, L)$ and $D = (s)$. We call

$$\lambda_D(x) = -\log \| s(x) \| \tag{5}$$

the Weil function, where $s \in H^0(M, L)$ and $D = (s)$. We can also define $N_f(D, r)$ in a similar way.

We consider a positive line bundle L on M and q divisors D_j of holomorphic sections s_j of the bundle such that

(A) $D_1, \cdots, D_q$ are manifolds intersect in general position;

(B) $qc_1(L) + c_1(K_M) > 0$ where K_M is the canonical bundle.

Theorem 5[5] (The SMT by Carlson and Griffiths) Let M be a projective variety of dimension n, and $f: \mathbf{C}^n \to M$ be a differentially non-degenerate holomorphic map. Let $D = \sum_{j=1}^{q} D_j$ be a divisor on M satisfying (A) and (B). Let K_M be the canonical line bundle over M. Then, for any $\delta > 0$,

$$m_f(r, D) + T_{f,K_M}(r) + N(S_f, r) \leqslant O(\log T_{f,L}(r)) + \delta \log r \,\|_{\delta}.$$

The proof is similar to the Riemann surface case above. We consider

$$\Psi = \frac{\Omega}{\prod_{j=1}^{q}(\|s_j\|^2(\log\|s_j\|^2)^2)}$$

where Ω is a volume form (a global positive (n,n) form on M). Write $f^*\Psi = \Gamma\Phi_\zeta$, where Φ_ζ is the Euclidean volume form in $\mathbf{C}^n$. Then, similar to the above, we can get

$$m_f(r, D) + T_{f,K_M}(r) + N(S_f, r) \leqslant O(\log \hat{T}(r)) + \delta \log r \,\|_{\delta},$$

where

$$\hat{T}(r) = \int_0^r \frac{\mathrm{d}t}{t^{2n-1}} \int_{B_t} \Gamma^{1/n} \phi_0^n.$$

Similar to the proof in the Riemann surface case, we need to bound $\hat{T}(r)$ in terms of $T_f(L, r)$. Instead of (2), Carlson and Griffiths proved the following claim:

Claim: (a) $\mathrm{Ric}\Psi > 0$; (b) $(\mathrm{Ric}\Psi)^n > \Psi$; (c) $\int_{M \backslash D} (\mathrm{Ric}\Psi)^n < \infty$.

We now use the claim to finish the proof. We show that

$$(f^* \mathrm{Ric}\Psi) \wedge \phi_0^{n-1} \geqslant c\Gamma^{1/n}\phi_0^n.$$

In fact, writing

$$f^*\mathrm{Ric}\Psi = \frac{\sqrt{-1}}{2\pi} \sum_{j,k=1}^{n} R_{j,k} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k.$$

Then, by the claim,

$$\Gamma\Phi_z = f^*\Psi \leqslant (f^*\mathrm{Ric}\Psi)^n = n! \det \mathrm{R} \, \Phi_z,$$

where $\mathrm{R} = (R_{jk})$. Use

$$\det(\mathrm{R})^{1/n} \leqslant \mathrm{Tr}\,\mathrm{R}/n,$$

so

$$\left(\frac{\Gamma}{n!}\right)^{1/n} \phi_0^n \leqslant \frac{1}{n} \sum_{j=1}^{n} R_{jj}\phi_0^n.$$

But $f^*(\mathrm{Ric}\Psi) \wedge \phi_0^{n-1} = n^{-1} \sum_{j=1}^{n} R_{jj}\phi_0^n$. Thus $(f^*\mathrm{Ric}\Psi) \wedge \phi_0^{n-1} \geqslant c\Gamma^{1/n}\phi_0^n$.

Now, applying the above inequality, we get $\hat{T}(r) \leqslant cT_{f,\mathrm{Ric}\Psi}(r)$ for some positive constant c, where

$$T_{f,\mathrm{Ric}\Psi}(r) = \int_0^r \frac{\mathrm{d}t}{t^{2n-1}} \int_{B_t} f^*(\mathrm{Ric}\Psi) \wedge \phi_0^{n-1}.$$

It remains to estimate $T_{f,\mathrm{Ric}\Psi}(r)$. From the definition,

$$\mathrm{Ric}\Psi = qc_1(L) + c_1(K_M) - \sum_{j=1}^{q} \mathrm{dd}^c \log(\log\|s_j\|^2)^2,$$

so $T_{f,\mathrm{Ric}\Psi}(r) \leqslant qT_{f,L}(r) + T_{f,K_M}(r)$.

Using the condition (B) that $qc_1(L) + c_1(K_M) > 0$, we get $T_{f,\mathrm{Ric}\Psi}(r) \leqslant O(T_{f,L}(r))$. This completes the proof of the Second Main Theorem.

Based on the above theorem, Griffiths made the following conjecture.

Conjecture 1 (Griffiths) Let M be a projective variety of dimension n. Let $f: \mathbf{C} \to M$ be a holomorphic map with Zariski-dense image. Let L be a positive line bundle and let D_j, $1 \leqslant j \leqslant q$, be the divisors of holomorphic sections s_j of L such that the conditions (A) and (B) holds. Let K_M be the canonical line bundle over M. Then

$$T_{f,L}(r) + T_{f,K_M}(r) \leqslant N_f^{(n)}(r, D) + O(\log T_{f,L}(r)) + \delta \log r \,\|_{\delta}.$$

We consider the case that $M = \mathbf{P}^n(\mathbf{C})$. To determine the canonical divisor we consider the differential form $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$ in the affine coordinates $(1, x_1, \cdots, x_n)$ on $U_0 = \{x \in \mathbf{P}^n \mid x_0 \neq 0\}$. There are no zeros or poles on U_0. But if we rewrite Ω with respect to $(x_0, \cdots, 1, \cdots, x_n)$ on $U_i = \{x \in \mathbf{P}^n \mid x_i \neq 0\}$, we find

$$\Omega = \frac{1}{x_0^{n+1}} \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}\hat{x}_i \wedge \cdots \wedge \mathrm{d}x_n.$$

Hence Ω has a pole of order $n+1$ along $x_0 = 0$ and $K_M = -(n+1)H$ where H is the hyperplane at infinity. So $T_{f,K_M}(r) = -(n+1)T_{f,\mathscr{O}_{\mathbf{P}^n(1)}}(r)$. Take $D = H_1 + \cdots + H_q$, where $H_1, \cdots, H_q$ are hyperplanes in general position, then conjecture is the theorem, known as Cartan's SMT for holomorphic curves intersecting hyperplanes. We discuss it in the next section.

3 H. Cartan's Result

In 1933, H. Cartan[6] extended Nevanlinna's theory to holomorphic mappings from $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$ intersecting hyperplanes, where $\mathbf{P}^n(\mathbf{C})$ is the n-dimensional complex projective space. To state Cartan's result, we rewrite the three Nevanlinna's functions in the case $M=\mathbf{P}^n(\mathbf{C})$. Let $\mathscr{O}_{\mathbf{P}^n}(1)$ be the hyperplane line bundle over $\mathbf{P}^n(\mathbf{C})$. The metric is given by $h_\alpha(z)=|z_\alpha|^2/\|z\|^2$ on $U_\alpha=\{z=[z_0:\cdots:z_n]\in\mathbf{P}^n\,|z_\alpha\neq 0\}$, where $\|z\|=\max\limits_{0\leqslant i\leqslant n}|z_i|$, so that $c_1(\mathscr{O}_{\mathbf{P}^n}(1))=\mathrm{dd}^c\log\|z\|^2=\omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbf{P}^n(\mathbf{C})$. Let $f=[f_0:\cdots:f_n]:\mathbf{C}\to\mathbf{P}^n(\mathbf{C})$ be a holomorphic map, where $f_0,\cdots,f_n$ are entire functions without common zeros. The Nevanlinna-Cartan height function $T_{f,\mathscr{O}_{\mathbf{P}^n}(1)}(r)$ (we also simply denote it by $T_f(r)$) is, from (3),

$$T_f(r)=T_{f,\mathscr{O}_{\mathbf{P}^n}(1)}(r)=\int_0^r\frac{\mathrm{d}t}{t}\int_{|z|<t}f^*\omega_{FS}.$$

Any holomorphic section s of $\mathscr{O}_{\mathbf{P}^n}(1)$ is given by $s([z_0:\cdots:z_n])=a_0z_0+\cdots+a_nz_n$ for some complex numbers $a_0,\cdots,a_n$ (i.e. $s=\{s_\alpha\}$, where $s_\alpha=(a_0z_0+\cdots+a_nz_n)/z_\alpha$ on $U_\alpha=\{[z_0:\cdots:z_n]\in\mathbf{P}^n\,|z_\alpha\neq 0\}$. Obviously $(s)=H=\{[z_0:\cdots:z_n]\in\mathbf{P}^n(\mathbf{C}):a_0z_0+\cdots+a_nz_n=0\}$ which is a hyperplane in $\mathbf{P}^n(\mathbf{C})$. Thus, the proximity function $m_f(r,H)$ is given by, from (4),

$$m_f(r,H)=-\int_0^{2\pi}\log\|s(f(re^{\mathrm{i}\theta}))\|\frac{\mathrm{d}\theta}{2\pi}=$$
$$\log\int_0^{2\pi}\log\frac{\|f(re^{\mathrm{i}\theta})\|\cdot\|L\|}{|L(f)(re^{\mathrm{i}\theta})|}\frac{\mathrm{d}\theta}{2\pi}+O(1),$$

where L is the linear form $L(x)=a_0x_0+\cdots+a_nx_n$, $\|L\|=\max\limits_{0\leqslant i\leqslant n}|a_i|$ and $\|f\|=\max\limits_{0\leqslant i\leqslant n}|f_i|$.

Note that the Weil function is $\lambda_H(x)=\log\frac{\|x\|\cdot\|L\|}{|L(x)|}$ for $x\in\mathbf{P}^n\backslash H$. The counting function $N_f(r,H)$ is defined as

$$N_f(r,H)=\int_{r_0}^r\frac{n_f(t,H)}{t}\mathrm{d}t,$$

where $r_0>0$ is fixed, $n_f(r,H)=\#$ of points in $|z|<r$ with $L(f)(z)=0$, counting multiplicities.

Theorem 6[6] (Cartan's SMT) Let $f:\mathbf{C}\to\mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate holomorphic map. Let $H_1,\cdots,H_q$ be hyperplanes in general position in $\mathbf{P}^n(\mathbf{C})$. Then, for any $\varepsilon>0$,

$$\sum_{j=1}^q m_f(r,H_j)\leqslant(n+1+\varepsilon)T_f(r)\ \|.$$

Note that Cartan actually obtained the stronger result: the SMT with truncations.

In our applications, we need the general form of H. Cartan's result where the "general position" condition for the given hyperplanes is dropped[7]. The new version is basically equivalent to the original version, but is much easier to use.

Theorem 7[7] (The General Cartan's Theorem) Let $f:\mathbf{C}\to\mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate holomorphic map. Let $H_1,\cdots,H_q$ (or linear forms $L_1,\cdots,L_q$) be arbitrary hyperplanes in $\mathbf{P}^n(\mathbf{C})$. Then, for every $\varepsilon>0$,

$$\int_0^{2\pi}\max_K\sum_{j\in K}\lambda_{H_j}(f(re^{\mathrm{i}\theta}))\frac{\mathrm{d}\theta}{2\pi}\leqslant(n+1+\varepsilon)T_f(r)\ \|$$

where the maximum is taken over all $K\subset\{1,\cdots,q\}$ such that the linear forms $L_j, j\in K$, are linearly independent.

For hyperplanes $H_1,\cdots,H_q$ in general position we have the following product to the sum estimate[7].

Lemma 1[7] (Product to the sum estimate) Let $H_1,\cdots,H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$, located in general position. Denote by T the set of all injective maps $\mu:\{0,1,\cdots,n\}\to\{1,\cdots,q\}$. Then

$$\sum_{j=1}^q m_f(r,H_j)\leqslant\int_0^{2\pi}\max_{\mu\in T}\sum_{i=0}^n\lambda_{H_{\mu(i)}}(f(re^{\mathrm{i}\theta}))\frac{\mathrm{d}\theta}{2\pi}+O(1).$$

Therefore, if $H_1,\cdots,H_q$ are in general position, then the general version easily implies H. Cartan's original theorem. The proof of the General Cartan's Theorem (as well as the original Cartan's theorem) uses the LDL stated above (see [7] for details).

In the rest of the paper, we discuss some recent developments in establishing Second Main Theorem for holomorphic curves into an arbitrary projective variety, extending the result of H. Cartan. The method indeed is motivated by the techniques from Diophantine approximation in number theory. In 2002, in the paper entitled "A subspace theorem approach to integral points on curves"[8], P. Corvaja and U. Zannier started the program of studying integral points on algebraic varieties by using Schmidt's subspace theorem in Diophantine approximation. Since then, the program has led a great progress in the study of Diophantine approximation[9-15]. It is known that the counterpart of Schmidt's subspace in Nevanlinna theory is H. Cartan's Second Main Theorem. In recent years, the method of P. Corvaja and U. Zannier has been adapted by a number of authors and a big progress has been made in extending the Second Main Theorem to holomorphic mappings from $\mathbf{C}$ into arbitrary projective variety X intersecting general divisors by using H. Cartan's original theorem [16-21]. We call such method "a Cartan's Second Main Theorem approach". We discuss this method in the next section.

4 Holomorphic Curves into Projective Varieties

In this section, we use the "Cartan's Second Main Theorem approach" to establish the Second Main Theorem for holomorphic curves into an arbitrary projective variety intersecting general divisors.

4.1 The Basic Theorem

The starting point is the following result which is basically a reformulation of H. Cartan's theorem stated above. We call it the "Basic Theorem". Its proof can be found in [21].

Theorem 8 (The Basic Theorem) Let X be a complex projective variety and let D be an effective Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X,\mathscr{O}(D))$, and let $s_1,\cdots,s_q$ be nonzero elements of V. Let $f:\mathbf{C}\to X$ be a holomorphic map with Zariski-dense image. Then, for any $\varepsilon>0$,

$$\int_0^{2\pi}\max_J\sum_{j\in J}\lambda_{s_j}(f(re^{i\theta}))\frac{d\theta}{2\pi}\leqslant(\dim V+\varepsilon)T_{f,D}(r)\ \|$$

where the set J ranges over all subsets of $\{1,\cdots,q\}$ such that the sections $(s_j)_{j\in J}$ are linearly independent.

4.2 The Nevanlinna Constant

The above Basic Theorem motivates the notion of the Nevanlinna constant introduced by the author [14,19]. Let X be a normal projective variety and D be an effective Cartier divisor on X. For any section $s\in H^0(X,\mathscr{O}(D))$ and prime divisor E on X, we use $\mathrm{ord}_E s$, or $\mathrm{ord}_E(s)$, to denote the coefficient of E in the divisor (s), where (s) is the divisor on X associated to s. We also sometimes call $\mathrm{ord}_E s$ the multiplicity of E in (s).

Definition 1 Let X be a complex projective variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. If X is normal, then we define

$$\mathrm{Nev}(\mathscr{L},D)=\inf_{N,V,\mu}(\dim V_N/\mu_N).$$

Here the inf is taken over all triples (N,V_N,μ_N) such that $N\in\mathbf{Z}>0$, V_N is a linear subspace of $H^0(X,\mathscr{L}^N)$ with $\dim V_N>1$, and $\mu_N>0$ is a positive real number, that satisfy the following property. For all $P\in\mathrm{Supp}\,D$, there exists a basis B of V_N with

$$\sum_{s\in B}\mathrm{ord}_E(s)\geqslant\mu_N\mathrm{ord}_E(ND)$$

for all irreducible component E of D passing through P. If there are no such triples (N,V,μ), then $\mathrm{Nev}(\mathscr{L},D)$ is defined to be $+\infty$. For a general projective variety X, $\mathrm{Nev}(\mathscr{L},D)$ is defined by pulling back to the normalization of X.

Note that, in [19], the Nevanlinna constant was only defined for $\mathscr{L}=\mathscr{O}_X(D)$, which is denoted by $\mathrm{Nev}(D):=\mathrm{Nev}(\mathscr{O}_X(D),D)$. The definition given above is indeed more general and would be potentially useful.

Theorem 9 [19] (Ru) Let X be a complex projective variety, let D be an effective Cartier divisor and $\mathscr{L}$ be a line sheaf on X with $\dim H^0(X,\mathscr{L}^N)\geqslant 1$ for some $N>0$. Then, for every $\varepsilon>0$,

$$m_f(r,D)\leqslant(\mathrm{Nev}(\mathscr{L},D)+\varepsilon)T_{f,\mathscr{L}}(r)\ \|$$

holds for any holomorphic mapping $f:\mathbf{C}\to X$ with Zariski-dense image.

The proof uses the Basic Theorem above together with the properties of Weil functions (see (5) for the definition of the Weil function) stated below.

Lemma 2 [19,22] The Weil functions λ_D for Cartier divisors D on a complex projective variety X satisfy the following properties.

(i) Additivity: If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1+\lambda_2$ extends uniquely to a Weil function for D_1+D_2.

(ii) Functoriality: If λ is a Weil function for a Cartier divisor D on X, and if $\phi:X'\to X$ is a morphism such that $\phi(X')\not\subset\mathrm{Supp}D$, then $x\mapsto\lambda(\phi(x))$ is a Weil function for the Cartier divisor ϕ^*D on X'.

(iii) Normalization: If $X = \mathbf{P}^n$, and if $D = \{z_0 = 0\} \subset X$ is the hyperplane at infinity, then the function

$$\lambda_D([z_0 : \cdots : z_n]) := \log \frac{\max\{|z_0|, \cdots, |z_n|\}}{|x_0|}$$

is a Weil function for D.

(iv) Uniqueness: If both λ_1 and λ_2 are Weil functions for a Cartier divisor D on X, then $\lambda_1 = \lambda_2 + O(1)$.

(v) Boundedness from below: If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.

(vi) Principal divisors: If D is a principal divisor (f), then $-\log|f|$ is a Weil function for D.

Outline of the proof of Theorem 9. Denote by σ_0 the set of all prime divisors occurring in D, so we can write

$$D = \sum_{E \in \sigma_0} \mathrm{ord}_E(D) E.$$

Let

$$\Sigma := \{\sigma \subset \sigma_0 \mid \bigcap_{E \in \sigma} E \neq \varnothing\}.$$

For an arbitrary $x \in X$, we can pick $\sigma \in \Sigma$ (depends on x) for which

$$\lambda_D(x) \leqslant \lambda_{D_{\sigma,1}}(x) + O(1),$$

where $D_{\sigma,1} := \sum_{E \in \sigma} \mathrm{ord}_E(D) E$. Now for each $\sigma \in \Sigma$, by definition, there is a basis B_σ of V_N is a linear subspace of $H^0(X, \mathscr{L}^N)$ such that

$$\sum_{s \in B_\sigma} \mathrm{ord}_E(s) \geqslant \mu_N \mathrm{ord}_E(ND),$$

at some (and hence all) points $P \in \bigcap_{E \in \sigma} E$. Since Σ is finite, $\{B_\sigma | \sigma \in \Sigma\}$ is a finite collection of bases of V_N. Thus, we have, using the property of Weil function (see (v) in Lemma 2) that, if $D_1 \geqslant D_2$, then $\lambda_{D_1} \geqslant \lambda_{D_2} + O(1)$, we get that,

$$\lambda_{ND}(x) \leqslant \frac{1}{\mu_N} \max_{\sigma \in \Sigma} \sum_{s \in B_\sigma} \lambda_s(x) + O(1).$$

The theorem can thus be derived by taking $x = f(re^{i\theta})$, by taking integration and then by applying the Basic Theorem above.

Corollary 1 Let D be an ample divisor on a complex projective variety X. If $\mathrm{Nev}(D) < 1$, then every holomorphic map $f: \mathbf{C} \to X \backslash D$ is not Zariski dense, i.e., the image of f must be contained in a proper subvariety of X.

We can derive the known Second Main Theorem type results by simply computing the Nevanlinna constant. We first provide the following example to see how to compute the Nevanlinna's constant.

Example 1 Let $X = \mathbf{P}^n$ and $D = H_1 + \cdots + H_q$ where $H_1, \cdots, H_q$ are hyperplanes in $\mathbf{P}^n$ in general position. We take $N = 1$ and consider $V_1 := H^0(\mathbf{P}^n, \mathscr{O}(D)) \cong H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(q))$. Then $\dim V_1 = \binom{q+n}{n}$.

For each $P \in \mathrm{Supp}\, D$, since $H_1, \cdots, H_q$ are in general position, $P \in H_{i_1} \cap \cdots \cap H_{i_l}$ with $\{i_1, \cdots, i_l\} \subset \{1, \cdots, q\}$ and $l \leqslant n$. Without loss of generality, we can just assume $H_{i_1} = \{z_1 = 0\}, \cdots, H_{i_l} = \{z_l = 0\}$ by taking proper coordinates for $\mathbf{P}^n$. Now we take the basis $B = \{z_0^{i_0}, \cdots, z_n^{i_n} | i_0 + \cdots + i_n = q\}$ for $V_1 = H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(q))$. Then, for each irreducible component E of D containing P, say $E = \{z_{j_0} = 0\}$ with $1 \leqslant j_0 \leqslant l$, we have $\mathrm{ord}_E\{z_j = 0\} = 0$ for $j \neq j_0$, $\mathrm{ord}_E\{z_{j_0} = 0\} = 1$ and thus $\mathrm{ord}_E D = 1$. On the other hand,

$$\sum_{s \in B} \mathrm{ord}_E s = \sum_{\vec{i}} i_{j_0} = \frac{1}{n+1} \sum_{\vec{i}} (i_0 + \cdots + i_n) = \frac{q}{n+1}\binom{q+n}{n} = \frac{q}{n+1}\dim V_1,$$

where, in above, the sum is taken for all $\vec{i} = (i_0, \cdots, i_n)$ with $i_0 + \cdots + i_n = q$, and we used the fact that the number of choices of $\vec{i} = (i_0, \cdots, i_n)$ with $i_0 + \cdots + i_n = q$ is $\binom{q+n}{n}$. Thus we can take $\mu_1 = \frac{q}{n+1}\dim V_1$, and hence,

$$\mathrm{Nev}(D) \leqslant \dim V_1 / \mu_1 = (n+1)/q.$$

The above example, together with Theorem 9, recovers the result of H. Cartan stated earlier under the slightly stronger assumption that "f is algebraically non-degenerate".

For $D = D_1 + \cdots + D_q$ where $D_1, \cdots, D_q$ are hypersurfaces of same degree in $\mathbf{P}^n$ in general position, similar to above, but using a more sophisticated "multi-index filtration" argument, we can also show that $\mathrm{Nev}(D) \leqslant (n+1)/q$. Thus Theorem 9 recovers the following earlier result.

Theorem 10[17] (Ru's SMT for hypersurfaces) Let $f: \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic map with Zariski-dense image. Let $D_1, \cdots, D_q$ be hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ of degree d_j, located in general position. Then, for every

$\varepsilon > 0$,

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leqslant (n + 1 + \varepsilon) T_f(r) \quad \|.$$

More generally, let X be a complex projective variety and let $D = D_1 + \cdots + D_q$ where $D_1, \cdots, D_q$ are effective Cartier divisors such that each D_j, $1 \leqslant j \leqslant q$, is linearly equivalent to A for some fixed ample divisor A on X. Then we can show that $\mathrm{Nev}(D) \leqslant (\dim X + 1)/q$. So, again, Theorem 9 recovers the following result.

Theorem 11 [18] (Ru) Let X be a smooth complex projective variety of dimension n. Let $D_1, \cdots, D_q$ be effective Cartier divisors such that each D_j, $1 \leqslant j \leqslant q$, is linearly equivalent to $d_j A$ for some positive integers d_j, where A is a fixed ample divisor on X. We also assume that $D_1, \cdots, D_q$ are in general position on X. Let $f: \mathbf{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for every $\varepsilon > 0$,

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leqslant (n + 1 + \varepsilon) T_{f,A}(r) \quad \|.$$

4.3 The Recent Result of Ru-Vojta

Using the filtration and overall method in [6], Ru Min and P. Vojta established the following SMT for holomorphic curves into an arbitrary algebraic variety intersecting general divisors on X. To state the result, we first give some definition. Let $\mathscr{L}$ be a line sheaf on X, we use $h^0(\mathscr{L})$ to denote $\dim H^0(X, \mathscr{L})$, and $\mathscr{L}(-D)$ to denote the sheaf $\mathscr{L} \otimes \mathscr{O}(-D)$ for a given divisor D on X.

Definition 2 Let $\mathscr{L}$ be a line sheaf and D be a nonzero effective Cartier divisor on a projective variety X. We define

$$\gamma(\mathscr{L}, D) := \limsup_{N} \frac{N h^0(\mathscr{L}^N)}{\sum_{m \geqslant 1} h^0(\mathscr{L}^N(-mD))},$$

where N passes over all positive integers such that $h^0(\mathscr{L}^N(-D)) \neq 0$. If no such N exists, then we define $\gamma(\mathscr{L}, D) = +\infty$ (Note that $|\mathscr{L}^N|$ does not have to be base point free).

Theorem 12 [21] (Ru-Vojta) Let X be a smooth projective variety and let $D_1, \cdots, D_q$ be effective Cartier divisors in general position on X. Let $\mathscr{L}$ be a line sheaf on X with $h^0(\mathscr{L}^N) \geqslant 1$ for N big enough. Let $f: \mathbf{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for every $\varepsilon > 0$,

$$\sum_{j=1}^{q} m_f(r, D_j) \leqslant \left(\max_{1 \leqslant j \leqslant q} \gamma(\mathscr{L}, D_j) + \varepsilon\right) T_{f,\mathscr{L}}(r) \quad \|.$$

An earlier result of the dimension 2 case is obtained by S. Hussein and Ru Min [23]. To use Theorem 12, we compute $\gamma(L, D_j)$ with $L = D := D_1 + \cdots + D_q$ where each D_j, $1 \leqslant j \leqslant q$, is linearly equivalent to a fixed ample divisor A on X. We write $h^0(D) := h^0(\mathscr{O}(D))$. By the Riemann-Roch theorem, with $n = \dim X$, we have

$$h^0(ND) = h^0(qNA) = \frac{(qN)^n A^n}{n!} + o(N^n)$$

and

$$h^0(ND - mD_j) = h^0((qN - m)A) = (qN - m)^n A^n / n! + o(N^n).$$

Thus

$$\sum_{m \geqslant 1} h^0(ND - mD_j) = \frac{A^n}{n!} \sum_{l=0}^{qN-1} l^n + o(N^{n+1}) = \frac{A^n (qN - 1)^{n+1}}{(n+1)!} + o(N^{n+1}).$$

Hence

$$\gamma(D, D_j) = \lim_{N \to \infty} \frac{N (qN)^n A^n / n! + o(N^{n+1})}{A^n (qN - 1)^{n+1} / (n+1)! + o(N^{n+1})} = (n+1)/q.$$

Thus Theorem 12 again implies Theorem 11 stated earlier.

Sketch of the proof of Theorem 12. Choose $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and positive integers N and b such that

$$\left(1 + \frac{n}{b}\right) \max_{1 \leqslant i \leqslant q} \frac{N(h^0(\mathscr{L}^N) + \varepsilon_2)}{\sum_{m \geqslant 1} h^0(\mathscr{L}^N(-mD_i))} < \max_{1 \leqslant i \leqslant q} \gamma(\mathscr{L}, D_j) + \varepsilon_1. \tag{6}$$

Let

$$\Sigma = \left\{\sigma \subseteq \{1, \cdots, q\} \;\middle|\; \bigcap_{j \in \sigma} \mathrm{Supp}\, D_j \neq \varnothing\right\}.$$

For $\sigma \in \Sigma$, let

$$\Delta_\sigma = \left\{\boldsymbol{a} = (a_i) \in \mathbf{N}^{\#\sigma} \;\middle|\; \sum_{i \in \sigma} a_i = b\right\}.$$

For $\boldsymbol{a} \in \Delta_\sigma$, one defines the ideal $\mathscr{I}(x)$ of $\mathscr{O}_X$ by

$$\mathscr{I}(x) = \sum_{\boldsymbol{b}} \mathscr{O}_X\left(-\sum_{i \in \sigma} b_i D_i\right) \tag{7}$$

where the sum is taken for all $\boldsymbol{b} \in \mathbf{N}^{\#\sigma}$ with $\sum_{i \in \sigma} a_i b_i \geqslant bx$. Let

$$\mathscr{F}(\sigma; \boldsymbol{a})_x = H^0(X, \mathscr{L}^N \otimes \mathscr{I}(x)),$$

which we regard as a subspace of $H^0(X, \mathscr{L}^N)$, and let

$$F(\sigma; \boldsymbol{a}) = \frac{1}{h^0(\mathscr{L}^N)} \int_0^{+\infty} (\dim \mathscr{F}(\sigma; \boldsymbol{a})_x)\, \mathrm{d}x.$$

By Theorem 3.6 in [21] (see also Proposition 4.14 in [16]), we have

$$F(\sigma;\boldsymbol{a}) \geqslant \min_{1\leqslant i\leqslant q}\left(\frac{1}{h^0(\mathscr{L}^N)}\sum_{m\geqslant 1}h^0(\mathscr{L}^N(-mD_i))\right).$$

For any nonzero $s \in H^0(X,\mathscr{L}^N)$, we also define

$$\mu_{\boldsymbol{a}}(s) = \sup\{x \in \mathbf{R}^+: s \in \mathscr{F}(\sigma;\boldsymbol{a})_x\}. \quad (8)$$

Let $\mathscr{B}_{\sigma;\boldsymbol{a}}$ be a basis of $H^0(X,\mathscr{L}^N)$ adapted to the above filtration $\{\mathscr{F}(\sigma;\boldsymbol{a})_x\}_{x\in\mathbf{R}^+}$. By Remark 6.6 in [21], $F(\sigma,\boldsymbol{a}) = \frac{1}{h^0(\mathscr{L}^N)}\sum_{s\in\mathscr{B}_{\sigma;\boldsymbol{a}}}\mu_{\boldsymbol{a}}(s)$. Hence

$$\sum_{s\in\mathscr{B}_{\sigma;\boldsymbol{a}}}\mu_{\boldsymbol{a}}(s) \geqslant \min_{1\leqslant i\leqslant q}\sum_{m\geqslant 1}h^0(\mathscr{L}^N(-mD_i)). \quad (9)$$

It is important to note that there are only fininitely many ordered pairs $(\sigma,\boldsymbol{a})$ with $\sigma \in \Sigma$ and $\boldsymbol{a} \in \Delta_\sigma$.

Let $\sigma \in \Sigma$, $\boldsymbol{a} \in \Delta_\sigma$, and $s \in H^0(X,\mathscr{L}^N)$ with $s \neq 0$. Since the divisors D_i are all effective, it suffices to use only the leading terms in (7). The union of the sets of leading terms as x ranges over the interval $[0,\mu_{\boldsymbol{a}}(s)]$ is finite, and each such $\boldsymbol{b}$ occurs in the sum (7) for a closed set of x. Therefore the supremum (8) is actually a maximum.

Similarly, we have

$$\mathscr{L}^N\otimes\mathscr{F}(\mu_{\boldsymbol{a}}(s)) = \sum_{\boldsymbol{b}\in K}\mathscr{L}^N(-\sum_{i\in\sigma}b_iD_i),$$

where $K = K_{\sigma,\boldsymbol{a},s}$ is the set of minimal elements of $\{\boldsymbol{b} \in \mathbf{N}^{\#\sigma}: \sum_{i\in\sigma}a_ib_i \geqslant b\mu_{\boldsymbol{a}}(s)\}$ relative to the product partial ordering on $\mathbf{N}^{\#\sigma}$. This set is finite, so we have, for any prime divisor E on X,

$$\operatorname{ord}_E(s) \geqslant \min_{\boldsymbol{b}\in K}\sum_{i\in\sigma}b_i\operatorname{ord}_E(D_i). \quad (10)$$

For a basis $\mathscr{B}$ of $H^0(X,\mathscr{L}^N)$, denote by $(\mathscr{B})$ the sum of the divisors (s) for all $s \in \mathscr{B}$. Let E be a prime divisor on X, and let $v, v_{\sigma;\boldsymbol{a}}, v_i (i = 1,2,\cdots,q)$ be the multiplicities of E in D, $(\mathscr{B}_{\sigma;\boldsymbol{a}})$ and D_i, respectively. We claim that we can find some $\boldsymbol{a}$ such that

$$v_{\sigma;\boldsymbol{a}} \geqslant \frac{b}{b+n}\left(\min_{1\leqslant i\leqslant q}\sum_{m=1}^{\infty}h^0(\mathscr{L}^N(-mD_i))\right)v. \quad (11)$$

If $v = 0$ then there is nothing to prove, so we assume that $v > 0$. For $i \in \sigma$, let

$$t_i = v_i/v. \quad (12)$$

Note that $v_i = 0$ for all $i \notin \sigma$, so $\sum_{i\in\sigma}v_i = \sum_{i=1}^{q}v_i = v$, hence $\sum_{i\in\sigma}t_i = 1$. From the assumption that $D_1,\cdots,D_q$ lie in general position, we have $\#\sigma \leqslant n$. Therefore $b \leqslant \sum_{i\in\sigma}\lfloor(b+n)t_i\rfloor \leqslant b+n$, and we may choose $\boldsymbol{a} = (a_i) \in \Delta_\sigma$ such that

$$t_i \geqslant a_i/(b+n) \quad \text{for all } i \in \sigma. \quad (13)$$

For any $s \in \mathscr{B}_{\sigma;\boldsymbol{a}}$, let v_s be the multiplicity of E in the divisor (s). Using (10)、(12) ~ (13), and $\sum_{i\in\sigma}a_ib_i \geqslant b\mu_{\boldsymbol{a}}(s)$, we get

$$v_s \geqslant \min_{\boldsymbol{b}\in K}\sum_{i\in\sigma}b_iv_i = (\min_{\boldsymbol{b}\in K}\sum_{i\in\sigma}b_it_i)v \geqslant$$
$$\left(\min_{\boldsymbol{b}\in K}\sum_{i\in\sigma}a_ib_i/(b+n)\right)v \geqslant \left(\frac{b}{b+n}\right)\mu_{\boldsymbol{a}}(s)v, \quad (14)$$

where the set $K = K_{\sigma,\boldsymbol{a},s}$ is as in (10). Combining (14) and (9) then gives

$$\frac{v_{\sigma,\boldsymbol{a}}}{v} = \frac{1}{v}\sum_{s\in\mathscr{B}_{\sigma;\boldsymbol{a}}}v_s \geqslant \frac{b}{b+n}\sum_{s\in\mathscr{B}_{\sigma;\boldsymbol{a}}}\mu_{\boldsymbol{a}}(s) \geqslant$$
$$\frac{b}{b+n}\min_{1\leqslant i\leqslant q}\sum_{m\geqslant 1}h^0(\mathscr{L}^N(-mD_i)),$$

which gives (11).

From (11), we get

$$(\mathscr{B}_{\sigma;\boldsymbol{a}}) \geqslant \frac{b}{b+n}\left(\min_{1\leqslant i\leqslant q}\sum_{m=1}^{\infty}h^0(\mathscr{L}^N(-mD_i))\right)D.$$

Hence, from the property of Weil functions (see Lemma 2), it gives

$$\lambda_{\mathscr{B}_{\sigma;\boldsymbol{a}}} + O(1) \geqslant \frac{b}{b+n}\left(\min_{1\leqslant i\leqslant q}\sum_{m=1}^{\infty}h^0(\mathscr{L}^N(-mD_i))\right)\lambda_D. \quad (15)$$

Write

$$\bigcup_{\sigma;\boldsymbol{a}}\mathscr{B}_{\sigma;\boldsymbol{a}} = \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_{T_1} = \{s_1,\cdots,s_{T_2}\}.$$

For each $i = 1,\cdots,T_1$, let $J_i \subseteq \{1,\cdots,T_2\}$ be the subset such that $\mathscr{B}_i = \{s_j: j \in J_i\}$. Then, by (15),

$$\frac{b}{b+n}\left(\min_{1\leqslant i\leqslant q}\sum_{m\geqslant 1}h^0(\mathscr{L}^N(-mD_i))\right)\lambda_D(x) \leqslant$$
$$\max_{1\leqslant i\leqslant T_1}\lambda_{\mathscr{B}_i}(x) + O(1) = \max_{1\leqslant i\leqslant T_1}\sum_{j\in J_i}\lambda_{s_j}(x) + O(1).$$

Hence, by taking $x = f(z)$ and taking the integration, we get

$$\frac{b}{b+n}\left(\min_{1\leqslant i\leqslant q}\sum_{m\geqslant 1}h^0(\mathscr{L}^N(-mD_i))\right)m_f(r,D) \leqslant$$
$$\int_0^{2\pi}\max_{1\leqslant i\leqslant T_1}\sum_{j\in J_i}\lambda_{s_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} + O(1). \quad (16)$$

On the other hand, by the general Cartan's Theorem with ε_2 in place of ε, we have

$$\int_0^{2\pi}\max_{1\leqslant i\leqslant T_1}\sum_{j\in J_i}\lambda_{s_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leqslant (h^0(\mathscr{L}^N) +$$
$$\varepsilon_2)T_{f,\mathscr{L}^N}(r) + O(1) \parallel. \quad (17)$$

Using $T_{f,\mathscr{L}^N}(r) = NT_{f,\mathscr{L}}(r)$, the theorem is proved by combining (6),(16) and (17).

5 References

[1] Nevanlinna R. Zur theorie der meromorphen funktionen [J]. Acta Mathematica, 1925, 46(1/2): 1-99.

[2] Weyl H. Meromorphic functions and analytic curves [M]. Princeton: Princeton Univ Press, 1943.

[3] Chern S S. Complex analytic mappings of Riemann surfaces I [J]. Amer J Math, 1960, 82(2): 323-337.

[4] Ru Min. Nevanlinna theory and its relation to diophantine approximation [M]. River Edge: World Scientific, 2001.

[5] Carlson J, Griffiths P. A defect relation for equidimensional holomorphic mappings between algebraic varieties [J]. Ann of Math, 1972, 95(3): 557-584.

[6] Cartan H. Sur les zeros des combinaisions linearires de p fonctions holomorpes donnees [J]. Mathematica: Cluj, 1933, 7: 80-103.

[7] Ru Min. On a general form of the second main theorem [J]. Trans Amer Math Soc, 1997, 349(12): 5093-5105.

[8] Corvaja P, Zannier U. A subspace theorem approach to integral points on curves [J]. Comptes Rendus Mathematique, 2002, 334(4): 267-271.

[9] Faltings G, Wüstholz G. Diophantine approximations on projective varieties [J]. Invent Math, 1994, 116 (1): 109-138.

[10] Corvaja P, Zannier U. On integral points on surfaces [J]. Ann of Math, 2004, 160(2): 705-726.

[11] Corvaja P, Zannier U. On a general Thue´s equation [J]. Amer J Math, 2004, 126(5): 1033-1055.

[12] Evertse J H, Ferretti R G. Diophantine inequalities on projective varieties [J]. Int Math Res Notices, 2002, 12 (25): 1295-1330.

[13] Evertse J H, Ferretti R G. A generalization of the subspace theorem with polynomials of higher degree [M]. New-York, Vienna: Springer, 2008, 16: 175-198.

s [14] Ru Min. A general Diophantine inequality [J]. Functions et Approximation, 2017, 56(2): 143-163.

[15] Ru Min, Wang Julie Tzu-Yueh. A subspace theorem for subvarieties [J]. Algebra and Number Theory, 2017, 11 (10): 2323-2337.

[16] Autissier P. On the nondensity of integral points [J]. Duke Math J, 2011, 158(1): 13-27.

[17] Ru Min. A defect relation for holomorphic curves intersecting hypersurfaces [J]. Amer J Math, 2004, 126(1): 215-226.

[18] Ru Min. Holomorphic curves into algebraic varieties [J]. Annals of Mathematics, 2009, 169(1): 255-267.

[19] Ru Min. A defect relation for holomorphic curves intersecting general divisors in projective varieties [J]. J Geometric Analysis, 2016, 26(4): 2751-2776.

[20] Levin A. Generalizations of Siegel´s and Picard´s theorems [J]. Ann of Math, 2009, 170 (2): 609-655.

[21] Ru Min, Vojta P. Birational Nevanlinna constant and its consequences [J]. Preprint, arXiv: 1608.05382 [math. NT].

[22] Vojta P. Diophantine approximation and value distribution theory [M]. Berlin: Springer-Verlag, 1987.

[23] Hussein S, Ru Min. A general defect relation and height inequality for divisors in subgeneral position [J]. to appear in Asian J of Math.

Nevanlinna 理论的最新进展

汝 敏

(休斯顿大学数学系,美国德州 休斯顿 77204)

摘要: R. Nevanlinna 在 Picard 定理和 Borel 定理基础上,发表了他的论文,并建立了一个以其名字命名的理论. 此后,Nevanlinna 理论已经成为在复分析、复几何和多复变函数的一个重要研究领域. 该文旨在回顾以往研究中的一些重要进展,并对 Nevanlinna 理论研究中最新进展进行了部分综述.

关键词: Nevanlinna 理论; 第一基本定理; 第二基本定理

(责任编辑: 王金莲)