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有限级复合整函数的超级与超型

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摘要:该文研究了复合整函数$f(g(z))$的超级与超型,其中$f(z)$与$g(z)$均为有限级整函数. 在一定条件下,对复合函数$f(g(z))$的超型进行了精确的估计并改善了一些原有结果.

关键词:复合整函数;级;型;超级;超型

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0　引言

假设$f(z)$和$g(z)$是有限级整函数. 许多作者研究了当$f(z)$和$g(z)$是有限级时复合函数$f(g(z))$的增长性[1-6];2009 年文献[7]研究了当$f(z)$和$g(z)$是有限迭代级时,复合函数$f(g(z))$的增长级. 至今为止,关于复合函数的型的文献还是很少,即使对于有限级整函数复合的型的结果也少. 本文主要是研究有限级复合函数的超型,还研究了$\log_2 M(r,f(g))$与$\log M(r,g)$的增长性比较.

本文使用在亚纯函数Nevanlinna值分布理论中的标准符号[8-10]. 用$\rho(f)$、$\mu(f)$分别表示$f(z)$的级与下级. 对所有的$r>0$,令$\exp_1 r=\mathrm{e}^r$,$\exp_2 r=\exp(\exp r)$,$\exp_0 r=r=\log_0 r$. 对所有的$r>1$,定义$\log_1 r=\log r$,$\log_2 r=\log(\log r)$. 定义集合$E\subset(0,+\infty)$的线测度为$mE=\int_E \mathrm{d}t$以及其对数测度为$m_lE=\int_E \mathrm{d}t/t$,下面回顾一下整函数级和型的定义.

定义 1[8-9]　整函数$f(z)$的增长级$\rho(f)$和下级$\mu(f)$分别定义为

$$\rho(f)=\varlimsup_{r\to\infty}\frac{\log_2 M(r,f)}{\log r}=\varlimsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\quad(1)$$

$$\mu(f)=\varliminf_{r\to\infty}\frac{\log_2 M(r,f)}{\log r}=\varliminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.\quad(2)$$

定义 2[8-9]　若整函数$f(z)$的增长级满足$0<\rho(f)<\infty$,则整函数$f(z)$的增长型$\tau(f)$为

$$\tau(f)=\varlimsup_{r\to\infty}\frac{\log M(r,f)}{\log r^{\rho(f)}}.$$

相似地,定义整函数$f(z)$的增长下型$\underline{\tau}(f)$为

$$\underline{\tau}(f)=\varliminf_{r\to\infty}\frac{\log M(r,f)}{\log r^{\rho(f)}}.$$

定义 3[10]　整函数$f(z)$的超级$\rho_2(f)$为

$$\rho_2(f)=\varlimsup_{r\to\infty}\frac{\log_3 M(r,f)}{\log r}=\varlimsup_{r\to\infty}\frac{\log_2 T(r,f)}{\log r}.$$

定义 4　若整函数$f(z)$的超级满足$0<\rho_2(f)<\infty$,则整函数$f(z)$的超型$\tau_2(f)$为

$$\tau_2(f)=\varlimsup_{r\to\infty}\frac{\log_2 M(r,f)}{r^{\rho_2(f)}}.$$

定义 5[11-12]　整函数$f(z)$的有限迭代增长级$\rho_p(f)$和有限迭代增长下级$\mu_p(f)$分别为

$$\rho_p(f)=\varlimsup_{r\to\infty}\frac{\log_{p+1} M(r,f)}{\log r}=\varlimsup_{r\to\infty}\frac{\log_p T(r,f)}{\log r},$$

$$\mu_p(f)=\varliminf_{r\to\infty}\frac{\log_{p+1} M(r,f)}{\log r}=\varliminf_{r\to\infty}\frac{\log_p T(r,f)}{\log r},$$

其中p为正整数,若当$p=1,2$时,定义5就分别变成定义 1 和定义 2.

文献[7]研究了当$f(z)$和$g(z)$是有限迭代级整函数时,复合函数$f(g(z))$的增长级,得到如下结果.

定理 A[7]　假设$f(z)$和$g(z)$是有限迭代级整函数且$i(f)=p$,$i(g)=q$,若$\mu_p(f)>0$,则$i(f(g))=p+q$,$\rho_{p+q}(f(g))=\rho_q(g)$.

定理 B[7]　假设$f(z)$和$g(z)$是有限迭代级整函数且满足$0<\rho_p(f)<\infty$和$0<\mu_q(g)\leqslant\rho_q(g)<\infty$,则$i(f(g))=p+q$,$\rho_{p+q}(f(g))=\rho_q(g)$.

定理 C[7]　假设$f(z)$和$g(z)$是有限迭代级整

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函数且满足$0 < \mu_p(f) \leqslant \rho_p(f) < \infty$ 和$0 < \mu_q(g) \leqslant \rho_q(g) < \infty$,则

$$\frac{\mu_q(g)}{\rho_q(g)} \leqslant \varliminf_{r\to\infty}\frac{\log_{p+q+1} M(r, f(g))}{\log_q M(r,g)} \leqslant 1 \leqslant \varlimsup_{r\to\infty}\frac{\log_{p+q+1} M(r,f(g))}{\log_q M(r,g)} \leqslant \frac{\rho_q(g)}{\mu_q(g)}.$$

注意对定理 A ~ 定理 C 的引用只考虑了 $p = q = 1$ 的情形.

1　引理

引理 1[1]　假设$f(z)$ 和 $g(z)$ 是整函数且满足 $g(0) = 0$. 设 α 满足 $0 < \alpha < 1$ 并令 $c(\alpha) = (1 - \alpha)^2/(4\alpha)$,则对于 $r > 0$,有

$$M(M(r,g), f) \geqslant M(r, f(g)) \geqslant M(c(\alpha)M(\alpha r,g), f).$$

特别地,当 $\alpha = 1/2$ 时,有

$$M(r, f(g)) \geqslant M(M(r/2,g)/8, f).$$

引理 2[13]　假设$f(z)$、$g(z)$ 均是整函数,则

$$M(r, f(g)) \geqslant M((1 - o(1))M(r,g), f)\ (r\to\infty, r \notin E_0),$$

其中 E_0 是关于 r 的有穷对数测度集合.

引理 3[14]　假设$f(z)$ 为有限级整函数并满足 $0 <\rho(f) = \rho < \infty$,且$0 < \tau(f) = \tau < \infty$,则 $\forall \varepsilon > 0$,存在一个对数测度为无穷的集合 $E_1 \subset (0, +\infty)$,使得对于所有的 $r \in E_1$,有

$$M(r, f) > \exp((\tau - \varepsilon)r^{\rho}).$$

2　定理及其证明

定理 1　假设$f(z)$,$g(z)$ 是整函数且满足 $0 < \mu(f) \leqslant \rho(f) < \infty$,$0 < \rho(g) < \infty$ 和$0 < \tau(g) < \infty$,则$\mu(f)\tau(g) \leqslant \tau_2(f(g)) \leqslant \rho(f)\tau(g)$.

证　由引理 1 知,$\forall \varepsilon > 0$ 和无穷大的 r,有

$$M(r, f(g)) \leqslant M(M(r,g), f) \leqslant \exp((M(r, g))^{\rho(f)+\varepsilon}).$$

从上式及函数型的定义有

$$M(r, f(g)) \leqslant \exp_2((\rho(f) + \varepsilon)(\tau(g) + \varepsilon)\cdot r^{\rho(g)}),$$

由定理 A 知$\rho_2(f(g)) = \rho(g)$,则

$$\tau_2(f(g)) = \varlimsup_{r\to\infty}\frac{\log_2 M(r, f(g))}{r^{\rho_2(f(g))}} = \varlimsup_{r\to\infty}\frac{\log_2 M(r, f(g))}{r^{\rho(g)}} \leqslant \rho(f)\tau(g). \qquad (1)$$

另一方面,由引理 1 和定义 2 知,存在序列 $\{r_n\}(r_n \to \infty)$,使得

$$M(r_n, f(g)) \geqslant M(c(\alpha)M(\alpha r_n,g), f) \geqslant \exp(c_1(\exp((\tau(g) - \varepsilon)(\alpha r_n)^{\rho(g)}))^{\mu(f)-\varepsilon}), \qquad (2)$$

其中$c_1 = c(\alpha)^{\mu(f)-\varepsilon}$是一个依赖于$\alpha(0 < \alpha < 1)$和 $\mu(f)$ 的常数. 因此令 $\alpha \to 1$,由(2) 式可得

$$\tau_2(f(g)) = \varlimsup_{r\to\infty}\frac{\log_2 M(r_n, f(g))}{r_n^{\rho(g)}} \geqslant \mu(f)\tau(g). \qquad (3)$$

由不等式(1)、(3) 知结论成立,定理 1 得证.

推论 1　假设$f(z)$、$g(z)$ 是有限级整函数且满足$0 <\mu(f) = \rho(f) < \infty$,$0 < \rho(g) < \infty$ 和$0 < \tau(g) <\infty$,则$\tau_2(f(g)) = \rho(f)\tau(g)$.

推论 2　假设$f(z)$、$g(z)$ 满足定理 1 的假设条件且$\tau(g) = 0$ 或 $+\infty$,则$\tau_2(f(g)) = 0$ 或 $+\infty$.

例 1　若$f(z) = \exp(z^3)$,$g(z) = \exp(4z)$,则有 $f(g) = \exp((\exp(4z))^3)$,$\mu(f) = \rho(f) = 3$, $\rho(g) = 1$,$\tau(g) = 4$ 且 $\rho_2(f(g)) = \rho(g) = 1$, $\tau_2(f(g)) = 3\tau(g) = 12$.

定理 2　假设$f(z)$、$g(z)$ 是有限级整函数且满足 $0 < \rho(f) < \infty$ 和 $0 < \mu(g) = \rho(g) < \infty$,则 $\rho_2(f(g)) = \rho(g)$ 且 $\rho(f)\underline{\tau}(g) \leqslant \tau_2(f(g)) \leqslant \rho(f)\tau(g)$.

证　由定理 B 可得$\rho_2(f(g)) = \rho(g)$,且由定义 1 及引理 1 得

$$M(r, f(g)) \leqslant M(M(r,g), f) \leqslant \exp_2((\rho(f) + \varepsilon)(\tau(g) + \varepsilon)r^{\rho(g)}).$$

然后由上式和定义 2 得

$$\tau_2(f(g)) \leqslant \rho(f)\tau(g).$$

另一方面,由引理 1 知,存在序列$\{r_n\}(r_n \to \infty)$,有

$$M(r_n, f(g)) \geqslant M(c(\alpha)M(\alpha r_n,g), f) \geqslant \exp(c_3(\exp((\underline{\tau}(g) - \varepsilon)(\alpha r_n)^{\rho(g)})^{\rho(f)-\varepsilon})), \qquad (4)$$

其中$c_3 = c(\alpha)^{\rho(f)-\varepsilon} > 0$是一个依赖于$\alpha$和$\rho(f)$ 的常数. 然后由(4) 式得

$$\tau_2(f(g)) \geqslant \rho(f)\underline{\tau}(g).$$

因此定理 2 得证.

定理 3　假设$f(z)$ 是整函数且满足$0 < \rho(f) < +\infty$,$0 < \tau(f) <+\infty$,设$g(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$ 是 1 个 $n(n \geqslant 1)$ 次多项式,则$\rho(f(g)) = n\rho(f)$ 且 $\tau(f(g)) = \tau(f)|a_n|^{\rho(f)}$.

证　设$g(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$,则 $\forall \varepsilon > 0$ 和无穷大的 r,有

$$(|a_n| - \varepsilon)r^n \leqslant M(r,g) \leqslant (|a_n| + \varepsilon)r^n.$$

由引理 1 知

$$M(r, f(g)) \leqslant M(M(r,g), f) \leqslant \exp_2((\tau(f) + \varepsilon)(M(r,g))^{\rho_2(f)+\varepsilon}) \leqslant \exp((\tau(f) + \varepsilon)((|a_n| +$$

$\varepsilon)r^n)^{\rho(f)})$. (5)

另一方面,由引理2、引理3及定义4知,存在序列$\{r_m\}\subset E_1\backslash E_0$,使得

$M(r_m,f(g))\geqslant M((1-o(1))M(r_m,g),f)\geqslant \exp((\tau(f)-\varepsilon)((1-o(1))(|a_n-\varepsilon|)r_m^n)^{\rho(f)})$, (6)

其中E_0是有限对数测度集合,E_1是无穷对数测度集合.

结合不等式(5)~(6)得$\rho(f(g))=n\rho(f)$和$\tau(f(g))=\tau(f)|a_n|^{\rho(f)}$.

例2 若$f(z)=\exp(3z^2)$,$g(z)=4z^n+\cdots(n\geqslant 1)$,则有$\rho(f)=2$,$\tau(f)=3$,$\rho(f(g))=2n$和$\tau(f(g))=3\times 4^2=48$.

定理4 假设$f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$是1个$n(n\geqslant 1)$次多项式,$g(z)$为整函数并满足$0<\rho(g)<+\infty$且$0<\tau(g)<+\infty$,则$\rho(f(g))=\rho(g)$,$\tau(f(g))=n\tau(g)$.

证 由于$f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$,$\forall\varepsilon>0$和足够大的$r$,有

$$(|a_n|-\varepsilon)r^n\leqslant M(r,f)\leqslant(|a_n|+\varepsilon)r^n.$$

由引理1,得

$M(r,f(g))\leqslant M(M(r,g),f)\leqslant(|a_n|+\varepsilon)(M(r,g))^n\leqslant(|a_n|+\varepsilon)(\exp((\tau(g)+\varepsilon)r^{\rho(g)}))^n=(|a_n|+\varepsilon)\exp(n((\tau(g)+\varepsilon)r^{\rho(g)}))$. (7)

另一方面,由引理2、引理3和定义4知,存在序列$\{r_m\}\subset E_1\backslash E_0$,使得

$M(r_m,f(g))\geqslant M(1-o(1)M(r_m,g),f)\geqslant(|a_n|-\varepsilon)((1-o(1))M(r_m,g))^n\geqslant(|a_n|-\varepsilon)(1-o(1))^n(\exp_2((\tau_2(g)-\varepsilon)r_m^{\rho_2(g)}))^n$. (8)

由(7)~(8)式可知有$\rho(f(g))=\rho(g)$,$\tau(f(g))=n\tau(g)$.

例3 若$f(z)=5z^n+\cdots(n\geqslant 1)$,$g(z)=\exp(4z^2)$,则有$\rho(g)=2$,$\tau(g)=4$,$\rho(f(g))=2$和$\tau(f(g))=n\tau(g)=4n$.

定理5 假设$f(z)$、$g(z)$为整函数且满足$0<\mu(f)\leqslant\rho(f)<\infty$,$0<\rho(g)<\infty$和$0<\underline{\tau}(g)\leqslant\tau(g)<\infty$,则有

$$\frac{\mu(f)\underline{\tau}(g)}{\tau(g)}\leqslant\varliminf_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\leqslant\rho(f),$$

$$\mu(f)\leqslant\varlimsup_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\leqslant\frac{\rho(f)\tau(g)}{\underline{\tau}(g)}. \quad (9)$$

若$\mu(f)=\rho(f)$,则

$$\frac{\rho(f)\underline{\tau}(g)}{\tau(g)}\leqslant\varliminf_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\leqslant\rho(f)\leqslant\varlimsup_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\leqslant\frac{\rho(f)\tau(g)}{\underline{\tau}(g)}. \quad (10)$$

证 首先,证明不等式(9).由定义2知,$\forall\varepsilon>0$和无穷大的r,有

$$\log M(r,g)\leqslant(\tau(g)+\varepsilon)r^{\rho(g)}. \quad (11)$$

由引理1、定义1和定义2知,$\forall\varepsilon>0$和无穷大的r,有

$M(r,f(g))\geqslant\exp(c_2(\exp(\underline{\tau}(g)-\varepsilon)\cdot(\alpha r)^{\rho(g)}))^{\mu(f)-\varepsilon})$, (12)

其中$c_2=(c(\alpha))^{\mu(f)-\varepsilon}>0$是一个依赖于$\alpha$和$\mu(f)$的常数.由不等式(12)知,$\forall\varepsilon>0$和无穷大的$r$,有

$\log_2 M(r,f(g))\geqslant(\mu(f)-\varepsilon)(\underline{\tau}(g)-\varepsilon)\cdot(\alpha r)^{\rho(g)}+O(1)$, (13)

其中$0<\alpha<1$.令$\varepsilon\to 0$,$\alpha\to 1$,由不等式(11)~(13)得

$$\varliminf_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\geqslant\frac{\mu(f)\underline{\tau}(g)}{\tau(g)}. \quad (14)$$

由定理1知$\tau_2(f(g))\leqslant\rho(f)\tau(g)$,然后由引理3和定义2知,$\forall\varepsilon>0$和无穷大的$r$,有

$$\log_2 M(r,f(g))\leqslant(\rho(f)\tau(g)+\varepsilon)r^{\rho(g)}, \quad (15)$$

$$\log M(r,g)\geqslant(\underline{\tau}(g)-\varepsilon)r^{\rho(g)}. \quad (16)$$

另一方面,仍然由定理1知$\tau_2(f(g))\geqslant\mu(f)\tau(g)$,且由定义2知,$\forall\varepsilon>0$,存在序列$\{r_n\}(r_n\to\infty)$(在(17)和(19)式中的$r_n$是不同的),使得

$$\log_2 M(r,g)\geqslant(\mu(f)\tau(g)-\varepsilon)r_n^{\rho(g)}, \quad (17)$$

$$\log M(r,g)\geqslant(\tau(g)-\varepsilon)r_n^{\rho(g)}. \quad (18)$$

令$\varepsilon\to 0$,由不等式(15)和(18)得

$$\varliminf_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\leqslant\rho(f). \quad (19)$$

令$\varepsilon\to 0$,由不等式(11)和(17)可知

$$\varlimsup_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\geqslant\mu(f). \quad (20)$$

令$\varepsilon\to 0$,由不等式(15)~(16)可知

$$\varlimsup_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}\leqslant\frac{\rho(f)\tau(g)}{\underline{\tau}(g)}. \quad (21)$$

因此由不等式(14)和不等式(19)~(21)知,不等式(9)成立.不等式(10)的证明只需将$\mu(f)=\rho(f)$代入不等式(9)即可.定理5得证.

推论3 若定理假设$f(z)$、$g(z)$是有限级整函数且满足$0<\mu(f)=\rho(f)<\infty$,$0<\rho(g)<\infty$和$0<\underline{\tau}(g)=\tau(g)<\infty$,则有

$$\lim_{r\to\infty}\frac{\log_2 M(r,f(g))}{\log M(r,g)}=\rho(f).$$

例 4　若 $f(z)=\exp(z^2)$,$g(z)=\exp(2z)$,则有 $\mu(f)=\rho(f)=2$,$\rho(g)=1$,$\underline{\tau}(g)=\tau(g)=2$ 和

$$\underline{\lim_{r\to\infty}}\frac{\log_2 M(r,f(g))}{\log M(r,g)}=\overline{\lim_{r\to\infty}}\frac{\log_2 M(r,f(g))}{\log M(r,g)}=1.$$

注 1　定理 5 是在型的条件下得出的结果,比定理 C 估计更精确.

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The Hyper-Order and Hyper-Type of Composite Entire Function of Finite Order

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Abstract: In this paper, the hyper order and type of composite entire function $f(g(z))$ are investigated, where $f(z)$ and $g(z)$ are entire functions of finite order. The hyper-type of composite function $f(g(z))$ are estimated under some conditions. The obtained results improve some previous results.

Key words: composite entire function; order; type; hyper order; hyper type

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