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Naturally Partial Orders and Locally Quasi-Adequate Semigroups

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Abstract: The naturally partial order $\leqslant_e$ on an abundant semigroup is defined. Some characterizations of $\leqslant_e$ are obtained. In particular, it is proved that for an abundant semigroup *S*, *S* is an idempotent-connected locally quasi-adequate semigroup if and only if $\leqslant = \leqslant_e$ on *S*. This enriches and extends the result of Lawson about locally orthodox semigroups.

Key words: naturally partial order; adequate semigroup; quasi-adequate semigroup; abundant semigroup

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0 Introduction

For a semigroup S and $e^2 = e \in S$, the subsemigroup eSe of S is a monoid. We call subsemigroup eSe a local monoid of *S*. S is called a locally P semigroup if all local monoids have the property *P*. Many classes of famous semigroups have some kinds of local properties, for example, completely simple semigroups are local groups, completely 0-simple semigroups are local 0-groups, etc. Nambooripad defined a naturally partial order $\leqslant$ on regular semigroups. Further, he established the connection between $\leqslant$ and locally inverse semigroups. That is, he proved that a regular semigroup S is a locally inverse semigroup if and only if with respect to $\leqslant$, S is an ordered semigroup[1]. In 1989, Lawson introduced a naturally partial order $\leqslant_e$ and verified that a regular semigroup is a locally orthodox semigroup if and only if $\leqslant_e = \leqslant$[2].

To generalize regular semigroups, Fountain[3] defined abundant semigroups. An abundant semigroup is defined as a semigroup in which each L^*-class and each R^*-class contains at least one idempotent. There are many authors having been studying various kinds of abundant semigroups, An abundant semigroup is called adequate[4] if all idempotents commute, an abundant semigroup is called quasi-adequate[5] if its set of idempotents forms a band. Inverse semigroups are adequate and orthodox semigroups are quasi-adequate. In 1987, Lawson defined three naturally partial orders $\leqslant_l$, $\leqslant_r$ and $\leqslant$ on abundant semigroups, which coincide with the Nambooripad order for regular semigroups. He pointed out that for an IC abundant semigroup whose set of regular elements form a subsemigroup, it is a locally adequate semigroup if and only if with respect to $\leqslant$, it is an ordered semigroup[6]. In [7], Guo Xiao-jiang and Luo Yan-feng proved that an IC abundant semigroup S is a locally adequate semigroup if and only if with respect to $\leqslant$, S is an ordered semigroup. Guo Xiao-jiang and K.P. Shum proved the properties of $\leqslant_l$ on a *rpp* semigroup[8]. For locally quasi-adequate semigroups, we have obtained some interesting results[9]. So we have a natural problem: whether locally quasiadequate semigroups have similar properties? In this note we shall consider this problem.

In this paper we shall use the notions and notations of [3]. Others can be found in Howie[10]. Here we list some known results used repeatedly in the sequel without mentions[11-15]. Firstly, we recall some basic facts about the relations L^* and R^*.

Lemma 1　The following statements are equiva-

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lent for a semigroup S and $a,b\in S$:

(i) $aL^*[R^*]b$,

(ii) For all $x,y\in S^1$, $ax=ay[xa=ya]$ if and only if $bx=by[xb=yb]$.

Corollary 1 Let S be a semigroup and $e^2=e$, $a\in S$, Then the following conditions are equivalent:

(i) $aL^*e[aR^*e]$,

(ii) $ae=a[ea=a]$ and for any $x,y\in S^1$, $ax=ay[xa=ya]$ implies that $ex=ey[xe=ye]$.

Evidently, L^* is a right congruence and R^* is a left congruence. In general, $L\subseteq L^*$ and $R\subseteq R^*$. But for regular elements a and b, $aL^*b[aR^*b]$ if and only if $aLb[aRb]$. For convenience, we use a^* to denote the idempotents L^*-related to a while a^+ those R^*-related to a. It is not difficult to see that in an adequate semigroup, each L^*-class and each R^*-class contains exactly one idempotent. Also, if K^* is one of Green's $*$-relations L^*, R^*, H^*, D^* and J^*, we denote by K_a^* the K^*-class of S containing a.

As in [16], an abundant semigroup S is idempotent-connected, in short, IC, if for each $a\in S$ and for some a^+, a^*, there exists a bijection θ : $\langle a^+\rangle\to\langle a^*\rangle$ such that $xa=a(x\theta)$ for all $x\in\langle a^+\rangle$, where $\langle e\rangle(e\in E(S))$ is the subsemigroup of S generated by the idempotents of eSe. In this case, θ is in fact an isomorphism.

Recall that ω is the natural order on the set of idempotents of semigroup T defined by: for $e,f\in E(T)$,

$$e\omega f \text{ if and only if } e=ef=fe.$$

In what follows, we denote the set $\{f\in E(T), f\omega e\}$ by $\omega(e)$.

Denoted by $R^*(x)$ [resp. $L^*(x)$] is the smallest right [resp. left] $*$-ideal containing x. We define $R_x^*\leqslant R_y^*$ if $R^*(x)\subseteq R^*(y)$ while $L_x^*\leqslant L_y^*$ if $L^*(x)\subseteq L^*(y)$. It is not difficult to check that these above relations are partial orders on S/R^* and S/L^*, respectively. Thus $xR^*[L^*]y$ if and only if $R_x^*\leqslant R_y^*$ and $R_y^*\leqslant R_x^*$ [$L_x^*\leqslant L_y^*$ and $L_y^*\leqslant L_x^*$]. As in [6], define on an abundant semigroups S: for $x,y\in S$, $x\leqslant_r y \Leftrightarrow R_x^*\leqslant R_y^*$ and $x=ey$ for some $e\in E(R_x^*)$, and $x\leqslant_l y \Leftrightarrow L_x^*\leqslant L_y^*$ and $x=yf$ for some $f\in E(L_x^*)$, and $\leqslant=\leqslant_l\cap\leqslant_r$. It is worth to pointing out that the restriction of $[\leqslant_r,\leqslant_l]\leqslant$ to $E(S)$ coincides with ω. Lawson noticed that $x\leqslant_r y$ [resp. $x\leqslant_l y$] if for each (some) idempotent $y^+\in R_y^*$ [resp. $y^*\in L_y^*$], there exists an idempotent $x^+\in R_x^*$ [resp. $x^*\in L_x^*$] such that $x^+\omega y^+$ [resp. $x^*\omega y^*$] and $x=x^+y$ [resp. $x=yx^*$].

Lemma 2 If S is an abundant semigroup, then

(i) For any idempotent e in S, eSe is an abundant subsemigroup of S,

(ii) S is idempotent-connected if and only if on S, $\leqslant_l=\leqslant=\leqslant_r$, if and only if for any $a\in S$,

(a) For some [for any] a^+, a^* and for any $e\in\omega(a^+)$, there exists $f\in\omega(a^*)$ such that $ea=af$,

(b) For some [for any] a^*; a^+ and for any $g\in\omega(a^*)$, there exists $h\in\omega(a^+)$ such that $ag=ha$.

For our aim, we need still the following lemma due to Hall[12].

Lemma 3 Let S be a semigroup and $e\in E(S)$. If $E(eSe)$ is a band, then both $E(eS)$ and $E(Se)$ are sub-bands of S.

1 Locally Quasi-Adequate Semigroup

Definition 1 Let S be an abundant semigroup and $a,b\in S$. Define $a\leqslant_e b\Leftrightarrow R_a^*\leqslant R_b^*, L_a^*\leqslant L_b^*$ and $a=ebf$ for some $e\in E(R_a^*), f\in E(L_a^*)$.

Lemma 4 Let S be an abundant semigroup. Then $\leqslant_e$ is a partial order on S such that

(i) $\omega=\leqslant_e\cap E(S)\times E(S)$,

(ii) On S, $\leqslant\subseteq\leqslant_l,\leqslant_r\subseteq\leqslant_e$.

Proof For $x\in S$, we have $R_x^*\leqslant R_x^*$, $L_x^*\leqslant L_x^*$. Note that S is abundant, there exist $e,f\in E(S)$ such that eR^*xL^*f and $x=exf$. By definition, $x\leqslant_e x$ and so $\leqslant_e$ is reflexive.

Assume $a,b\in S$ such that $a\leqslant_e b$ and $b\leqslant_e a$. By definition, $R_a^*\leqslant R_b^*$ and $R_b^*\leqslant R_a^*$, hence $R_a^*=R_b^*$, that is, aR^*b, dually, aL^*b. On the other hand, by definition, $a\leqslant_e b$ implies that $a=ebf$ for some $e\in E(R_a^*)$ and $f\in E(L_a^*)$. By the forgoing proofs, bR^*aR^*e and bL^*aL^*f . This shows that $b=ebf(=a)$. Thus $\leqslant_e$ is antisymmetric. Now let $a,b,c\in S$ with $a\leqslant_e b$ and $b\leqslant_e c$. By definition, $R_a^*\leqslant R_b^*\leqslant R_c^*$, $L_a^*\leqslant L_b^*\leqslant L_c^*$ and $a=ebf, b=gch$, where $e\in E(R_a^*)$, $g\in E(R_b^*)$, $f\in E(L_a^*)$ and $h\in E(L_b^*)$, hence $R_a^*\leqslant R_c^*, L_a^*\leqslant L_c^*$ and $a=(eg)c(hf)$. Since $R_e^*=R_a^*\leqslant R_c^*=R_g^*$, we have $eS=R^*(e)\subseteq R^*(g)=gS$, and $e=gx=g(gx)=ge(x\in S)$, hence $eg\in E(R_e^*)(=E(R_a^*))$, similarly, $hf\in E(L_a^*)$. Therefore $a\leqslant_e c$. We have now proved that $\leqslant_e$ is a partial order on S.

If $e,f\in E(S)$ and $e\leqslant_e f$, then $R_e^*\leqslant R_f^*$ and $L_e^*\leqslant L_f^*$. By the first inequality, $eS=R^*(e)\leqslant R^*(f)=fS$, thereby $e=fu=f(fu)=fe(u\in S)$, similarly, $e=ef$. Thus $e\omega f$, and whence $\leqslant_e\bigcap E(S)\times E(S)\subseteq\omega$. Note that $\omega\subseteq\leqslant_e\bigcap E(S)\times E(S)$. Therefore $\omega=\leqslant_e\bigcap E(S)\times E(S)$.

It remains to verify (ii). Since $\leqslant=\leqslant_l\bigcap\leqslant_r$ and $\leqslant_l$ is a dual of $\leqslant_r$, it suffices to show that $\leqslant_l\subseteq\leqslant_e$. To the end, let $u,v\in S$ and $u\leqslant_l v$. By definition, $L_u^*\leqslant L_v^*$ and $u=vf$ for some $f\in E(L_u^*)$. But $u=vf\in R^*(v)$, now $R_u^*\leqslant R_v^*$. On the other hand, $u=gu=gvf$ for any $g\in E(R_u^*)$. Thus $u\leqslant_e v$ and consequently, $\leqslant_l\subseteq\leqslant_e$. We complete the proof.

Let S be a semigroup and $\prec$ a partial order on S . We call $\prec$ preserves idempotents if for any $u,v^2=v\in S$, $u\prec v$ implies that u is an idempotent. And, $\prec$ is call to preserve the regularity if for any $u\in S$ and $v\in \mathrm{Re}\,g(S)$ (the set of regular elements of S), $u\prec v$ implies that u is regular. By [10], we have that for an abundant semigroup, $\leqslant$ preserves both idempotents and the regularity condition. It is a natural problem whether $\leqslant_e$ preserves idempotents (the regularity) or not? We proceed to answer this problem.

In general, we do not know whether $\leqslant_e$ preserves the regularity. But we can prove the following weaker result.

Theorem 1 If S is a locally quasi-adequate abundant semigroup, then $\leqslant_e$ preserves the regularity.

Proof Assume S is a locally quasi-adequate semigroup. Let $u,v\in S$ and $u\leqslant_e v$, then $R_u^*\leqslant R_v^*$, $L_u^*\leqslant L_v^*$ and $u=evf$ for some $e\in E(R_u^*)$, $f\in E(L_u^*)$. Since S is an abundant semigroup, there exist $g,h\in E(S)$ such that gR^*vL^*h . Note that $R_e^*=R_u^*\leqslant R_v^*=R_g^*$, we have $eS=R^*(e)=R^*(u)\subseteq R^*(v)=R^*(g)=gS$, so $e\in gS$, thereby $e=ge$, thus $eg\in E(S), eg\omega g$ and $egRe$. Similarly, $hf\in E(S)$, $hf\omega h$ and $hfLf$. Now assume v is a regular element of S and choose that v' is an inverse of v such that $h=v'v$ and $g=vv'$. It is not difficult to see that $v\cdot hf\cdot v'\in E(S)$ and $v\cdot hf\cdot v'\omega g$. By Lemma 3, these show that $eg\cdot v\cdot hf\cdot v'$ is an idempotent of gSg since $vhfv', eg\in E(gSg)$. Clearly, $egvhfv'\omega g$, hence $u=evf=egvhfv'\cdot v$ $\leqslant_r v$, thereby u is regular since $\leqslant_r$ preserves the regularity. Therefore $\leqslant_e$ preserves the regularity.

It is worth to mentioning that the converse of Theorem 1 is not true. This can follows from the fact: on a regular semigroup, $\leqslant_e$ always preserves the regularity.

Theorem 2 If S is an abundant semigroup, then S is a locally quasi-adequate semigroup if and only if $\leqslant_e$ preserves idempotents.

Proof Assume that S is a locally quasi-adequate semigroup. With notations in the proof of Theorem 1, we have $u\leqslant_r v$. If v is an idempotent, then u is an idempotent since $\leqslant_r$ preserves idempotents, whence $\leqslant_e$ preserves idempotents.

Conversely, suppose that $\leqslant_e$ preserves idempotents. Let $e\in E(S)$ and assume $x,y\in E(eSe)$. Obviously, $x=exe\omega e$ and $y\omega e$. These show that $R_x^*\leqslant R_e^*$ and $L_x^*\leqslant L_e^*$. Note that $xy=exe\cdot e\cdot eye$. Therefore $xy\leqslant_e e$, and whence xy is an idempotent of eSe since $\leqslant_e$ preserves idempotents. Consequently, $E(eSe)$ is a band. Again by Lemma 2, eSe is a quasi-adequate semigroup. We complete the proof.

Theorem 3 Let S be an abundant semigroup. Then S is an idempotent-connected semigroup which is locally quasi-adequate if and only if $\leqslant=\leqslant_e$.

Proof Assume S is an idempotent-connected semigroup which is locally quasi-adequate. Let $a, b\in S$ and $a\leqslant_e b$. By the proof of Theorem 1, for all $g\in E(R_b^*)$ and $h\in E(L_b^*)$, there are $e\in E(R_a^*)$, $f\in E(L_a^*)$ such that $e\omega g, f\omega h$ and $a=ebf$. Since

S is idempotent-connected and by Lemma 2, there is $k \in E(gSg)$ such that $bf = kb$. Note that $E(gSg)$ is a band, we observe that $ek \in E(S)$. Similarly, there exists $l \in E(S)$ such that $a = bl$. We have now proved that $a = ek \cdot b = bl$. In other words, $a \leqslant b$. Therefore $\leqslant_e \subseteq \leqslant$ and the reverse inclusion follows from Lemma 4.

Conversely, if $\leqslant_e = \leqslant$ on S, then $\leqslant_e$ preserves the regularity since $\leqslant$ preserves the regularity, hence by Theorem 2, S is a locally quasi-adequate semigroup. On the other hand, by Lemma 4, $\leqslant_l = \leqslant = \leqslant_r$ and whence S is idempotent-connected. This completes the proof.

It is well known that any regular semigroup is an idempotent-connected abundant semigroup. By Theorem 3, the following corollary is immediate, which is just the main result of [6].

Corollary 2 Let S be a regular semigroup. Then S is a locally orthodox semi-group if and only if $\leqslant = \leqslant_e$.

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自然偏序和局部拟适当半群

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摘要: 定义了富足半群上一个自然偏序 $\leqslant_e$，给出研究了自然偏序 $\leqslant_e$ 的性质，证明了：富足半群 S 是幂等元连通的局部拟适当半群当且仅当 $\leqslant = \leqslant_e$，丰富和推广了 Lawson 的局部半群的相关结果.

关键词：自然偏序；适当半群；拟适当半群；富足半群

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