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# 复模糊集值复模糊积分及其收敛性定理

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**摘要**: 首先介绍复模糊集值测度与复模糊集值可测函数的概念及复模糊集值可测函数的性质,以及基于复模糊集值复模糊测度的复模糊集值积分概念及其基本性质;其次,研究了复模糊集值复模糊积分的收敛问题,得到了这种拓广到复模糊集值上的复模糊积分的单调收敛定理、法都定理、控制收敛定理等重要的收敛性定理.

**关键词**: 复模糊集值测度; 复模糊集值可测函数; 复模糊集值复模糊积分; 收敛性定理

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## 0 引言

1998 年吴从炘等[1]将模糊测度的值域推广到模糊实数域,并定义了模糊数模糊测度的 Sugeno 积分;郭彩梅等[2-3]也定义模糊值函数关于模糊值测度的(G)积分,将关于模糊值模糊测度的 Sugeno 积分推广到模糊集上. 1989 年 J. J. Buckley[4]提出模糊复数概念,需要考虑模糊复数的测度与积分问题;张广全[5-9]引进取值于模糊实数的模糊距离讨论了模糊集上的模糊实数值测度问题,给出模糊(实)值模糊积分;1996 年仇计清等[10]将模糊测度与可测函数概念扩展到模糊复数集上给出复模糊测度、复模糊可测函数与复模糊积分概念;1999 年王贵君等[11]在 J. J. Buckley 的模糊复数概念基础上,给出了模糊复值测度与模糊复值积分概念,取得一些重要结论. 文献[12-15]研究了复模糊数集上的可测函数及其积分问题,特别研究了 Sugeno 型、Choquet 型模糊复数值积分及其性质,并将其应用于在分类技术中. 为了将模糊测度、模糊可测函数概念扩展到更广泛的复模糊集上,文献[16-18]研究了复模糊集值测度、复模糊集值可测函数与复模糊集值复模糊积分概念及其性质,本文在此基础上研究复模糊集值复模糊积分的收敛定理,拓展模糊测度与模糊积分理论.

## 1 预备

### 1.1 模糊实数与模糊复数

**定义 1**　称实数集 $\mathbf{R}$ 上的满足以下条件的映射 $\tilde{a}: \mathbf{R}^{+} \to [0,1]$ 为模糊实数(简称模糊数):

(i) $\tilde{a}$ 是正规的,即 $\exists x_0 \in \mathbf{R}$,使得 $\tilde{a}(x_0) = 1$;

(ii) $\forall \lambda \in (0,1]$,$\tilde{a}$ 的 $\lambda$ 截集 $a_\lambda = \{x \in \mathbf{R}: \tilde{a}(x) \geqslant \lambda\}$ 是有界闭区间,即 $a_\lambda = [a_\lambda^-, a_\lambda^+]$.

$\forall a \in \mathbf{R}$,规定

$$a(x) = \begin{cases} 1, x = a, \\ 0, x \neq a, \end{cases}$$

则 $\forall \lambda \in (0,1]$,$a_\lambda = \{a\} = [a,a]$,这说明模糊实数是普通实数的推广.

用 $F^*(\mathbf{R})$ 表示 $\mathbf{R}$ 上模糊数的全体,$F_+^*(\mathbf{R}) = \{\tilde{a} \in F^*(\mathbf{R}): 当 x \leqslant 0 时 \tilde{a}(x) = 0\}$ 表示正模糊数的全体,记 $\Delta(\mathbf{R})$ 为 $\mathbf{R}$ 上区间数的全体,$F(X)$ 表示集合 $X$ 上模糊集的全体,$\mathscr{F}$ 表示 $X$ 上的某个 $\sigma$-代数,从而 $(X,\mathscr{F})$ 是 1 个可测空间[19].

**定义 2**　设 $\mathbf{C}$ 为复数集,$\tilde{a},\tilde{b} \in F^*(\mathbf{R})$,定义映射 $\tilde{a} + \mathrm{i}\tilde{b}: \mathbf{C} \to [0,1]$,$(\tilde{a} + \mathrm{i}\tilde{b})(x + \mathrm{i}y) = \tilde{a}(x) \wedge \tilde{b}(y)$ $(\forall x + \mathrm{i}y \in \mathbf{C})$,称 $\tilde{a} + \mathrm{i}\tilde{b}$ 为 1 个模糊复数(或 Fuzzy 复数),称 $\tilde{a}$ 为 $\tilde{a} + \mathrm{i}\tilde{b}$ 的实部(记为 $\tilde{a} = \mathrm{Re}(\tilde{a} + \mathrm{i}\tilde{b})$),称 $\tilde{b}$ 为 $\tilde{a} + \mathrm{i}\tilde{b}$ 的虚部(记为 $\tilde{b} = \mathrm{Im}(\tilde{a} + \mathrm{i}\tilde{b})$).

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记 $c' = (\tilde{a},\tilde{b})$，则 $\tilde{a} = \mathrm{Re}\, c'$，$\tilde{b} = \mathrm{Im}\, c'$，特别地，当 $\tilde{b} = 0$ 时，规定 $(\tilde{a},\tilde{0}) = \tilde{a}$，$\forall a + \mathrm{i}b \in \mathbf{C}$，规定

$$c(z) = \begin{cases} 1, x = a, y = b, \\ 0, x \neq a \text{ 或 } y \neq b, \end{cases}$$

其中 $z = x + \mathrm{i}y$，这说明模糊复数是复数的推广，也是模糊实数的推广. 当 $A$ 和 $B$ 为 $\mathbf{R}$ 上的 2 个区间数时，称 $A + \mathrm{i}B$ 为 $\mathbf{C}$ 上的复区间数，$\mathbf{C}$ 上复区间数的全体记作 $\Delta(\mathbf{C})$.

模糊复数可以看成是复数和模糊实数的推广. 用 $F^*(\mathbf{C})$ 表示 $\mathbf{C}$ 上模糊复数的全体，$\mathscr{F}(\mathbf{C})$ 表示由复数集 $\mathbf{C}$ 的若干子集构成的 $\sigma$-代数. 记 $F_+^*(\mathbf{C}) = \{\tilde{A} + \mathrm{i}\tilde{B}: \tilde{A}, \tilde{B} \in F_+^*(\mathbf{R})\}$，用 $F_0(\mathbf{C}) = \{\tilde{X} + \mathrm{i}\tilde{Y}: \tilde{X}, \tilde{Y} \in F^*(\mathbf{R})\}$ 表示 $\mathbf{C}$ 上的有界闭模糊复数的全体.

对于 $\mathbf{R}$ 的给定子集 $A, B$，定义 $(A,B) = \{x + \mathrm{i}y: x \in A, y \in B\}$；对于 $F^*(\mathbf{C})$ 的 2 个成员 $A + \mathrm{i}B$ 和 $E + \mathrm{i}D$ 以及 1 个复数 $c = a + \mathrm{i}b \in \mathbf{C}$，定义运算 $\circ \in \{+, -, \cdot\}$ 如下：

$(A + \mathrm{i}B) \circ (E + \mathrm{i}D) = (A \circ E) + \mathrm{i}(B \circ D)$，

$c \circ (A + \mathrm{i}B) = cA + \mathrm{i}cB$.

称 $\tilde{a}$ 为模糊无穷大[8] 是指对于 $\tilde{a} \in F^*(\mathbf{R})$，若 $\forall M > 0, \exists \lambda_0 \in (0,1]$，s.t. $a_{\lambda_0}^+ > M$ 或 $a_{\lambda_0}^- < -M$，记为 $\tilde{\infty}$；对于 $c' \in F^*(\mathbf{C})$，若 $\tilde{a} = \mathrm{Re}\, c'$，$\tilde{b} = \mathrm{Im}\, c'$ 中至少 1 个为 $\tilde{\infty}$，称 $c'$ 是模糊无穷大，记为 $\tilde{\infty}$.

**约定 1**　$\forall c'_1, c'_2 \in F^*(\mathbf{C})$，$c'_1 \leqslant c'_2$ 当且仅当 $\mathrm{Re}\, c'_1 \leqslant \mathrm{Re}\, c'_2$，$\mathrm{Im}\, c'_1 \leqslant \mathrm{Im}\, c'_2$；$c'_1 = c'_2$ 当且仅当 $c'_1 \leqslant c'_2$，$c'_2 \leqslant c'_1$. $c'_1 < c'_2$ 当且仅当 $c'_1 \leqslant c'_2$ 且 $\mathrm{Re}\, c'_1 \leqslant \mathrm{Re}\, c'_2$ 或 $\mathrm{Im}\, c'_1 < \mathrm{Im}\, c'_2$；$c' \geqslant 0$ 为 $\mathrm{Re}\, c' \geqslant 0$，$\mathrm{Im}\, c' \geqslant 0$.

**约定 2**　设 $F^*(\mathbf{C})$ 表示 $\mathbf{C}$ 上全体 Fuzzy 复数，把满足下列条件的 $F^*(\mathbf{C})$ 的子集类记为 $A^*$：

(i) $\forall \tilde{A} \in A^*$，如果 $\tilde{B} = \{\inf \tilde{A}_0: \tilde{A}_0 \subset \tilde{A}\}$ 有上界，则 $\sup \tilde{B} \in F^*(\mathbf{C})$；

(ii) $\forall \tilde{A} \in A^*$，如果 $\tilde{C} = \{\sup \tilde{A}_0: \tilde{A}_0 \subset \tilde{A}\}$ 有下界，则 $\sup \tilde{B} \in F^*(\mathbf{C})$.

显然 $A^*$ 非空.

### 1.2　复模糊集值复模糊测度与复模糊集值复模糊可测函数

**定义 3**　设 $\tilde{\mu}: F(Z) \to F_+^*(\mathbf{C}) = \{\tilde{A} + \mathrm{i}\tilde{B}: \tilde{A}, \tilde{B} \in F_+^*(\mathbf{R})\}$，称 $\tilde{\mu}$ 是零可加(记为 0-add)，当且仅当若 $\forall \tilde{A}, \tilde{B} \in F(Z)$，$\tilde{A} \cup \tilde{B} \in F(Z)$，$\tilde{\mu}(\tilde{B}) = 0$，有 $\tilde{\mu}(\tilde{A} \cup \tilde{B}) = \tilde{\mu}(\tilde{A})$；称 $\tilde{\mu}$ 是零可减(记为 0-sub)，当且仅当若 $\forall \tilde{A}, \tilde{B} \in F(Z)$，$\tilde{A} \cap \tilde{B}^C \in F(Z)$，$\tilde{\mu}(\tilde{B}) = \tilde{0}$，有 $\tilde{\mu}(\tilde{A} \cap \tilde{B}^C) = \tilde{\mu}(\tilde{A})$；称 $\tilde{\mu}$ 上自连续(记为 autoc↓)，当且仅当若 $\forall \{\tilde{A}_n\} \subset F(Z)$，$\tilde{B} \in F(Z)$，$\tilde{A}_n \cup \tilde{B} \in F(Z)$，且 $(\tilde{\rho}) \lim\limits_{n\to\infty} \tilde{\mu}(\tilde{A}_n) = \tilde{0}$，有 $(\tilde{\rho}) \lim\limits_{n\to\infty} \tilde{\mu}(\tilde{A}_n \cup \tilde{B}) = \tilde{\mu}(\tilde{B})$；称 $\tilde{\mu}$ 下自连续(记为 autoc↑)，当且仅当若 $\forall \{\tilde{A}_n\} \subset F(Z)$，$\tilde{B} \in F(Z)$，$\tilde{A}_n^C \cap \tilde{B} \in F(Z)$，且 $(\tilde{\rho}) \lim\limits_{n\to\infty} \tilde{\mu}(\tilde{A}_n) = \tilde{0}$；有 $(\tilde{\rho}) \lim\limits_{n\to\infty} \tilde{\mu}(\tilde{A}_n \cap \tilde{B}) = \tilde{\mu}(\tilde{B})$；称 $\tilde{\mu}$ 自连续(记为 autoc)，当且仅当 $\tilde{\mu}$ 既上自连续又下自连续.

关于模糊实值可测函数、复模糊集值复模糊测度和复模糊值复模糊可测函数的概念可以参见文献[15-17].

记 $F_{(\alpha,\beta),\lambda} \triangleq \{z = \alpha + \mathrm{i}\beta: \mathrm{Re}f_\lambda^{\pm}(z) \geqslant \alpha, \mathrm{Im}f_\lambda^{\pm}(z) \geqslant \beta\}$，其中 $\mathrm{Re}f_\lambda^{\pm}(z) \geqslant \alpha$ 意指 $\mathrm{Re}f_\lambda^{+}(z) \geqslant \alpha$ 且 $\mathrm{Re}f_\lambda^{-}(z) \geqslant \alpha$，$\mathrm{Im}f_\lambda^{\pm}(z) \geqslant \beta$ 意指 $\mathrm{Im}f_\lambda^{+}(z) \geqslant \beta$ 且 $\mathrm{Im}f_\lambda^{-}(z) \geqslant \beta$，$\forall \alpha, \beta \in [0, \infty)$. 从而，$\tilde{f}$ 是 $\tilde{E}$ 上关于 $(Z, \mathscr{F}(Z), \tilde{\mu})$ 的复模糊值复模糊可测函数，当且仅当 $\forall \lambda \in [0, 1]$，$\tilde{E} \cap \chi_{F_{(\alpha,\beta),\lambda}} \in \mathscr{F}(Z)$，$\tilde{E} \cap \chi_{F_{(\alpha,\beta),\lambda}}^c \in \mathscr{F}(Z)$，用 $\tilde{M}(\tilde{E})$ 表示 $\tilde{E}$ 上复模糊集值复模糊可测函数全体.

复模糊集值可测函数的有关性质请参见文献[17]，复模糊集值复模糊积分的概念及有关性质请参见文献[18].

## 2　复模糊集值复模糊积分的收敛性定理

**引理 1**[12]　设 $\{f^{(n)}\}$ 为收敛的区间值函数序列，且 $\lim\limits_{n\to\infty} f^{(n)}$ 也是区间值函数，则

$$\lim_{n\to\infty} f^{(n)} = [\lim_{n\to\infty} f^{(n)-}, \lim_{n\to\infty} f^{(n)+}].$$

**引理 2**[12]　设 $\{\tilde{g}^{(n)}\}$ 为收敛的区间值函数列，则 $\lim\limits_{n\to\infty} \tilde{g}^{(n)} = \tilde{g} \Leftrightarrow \forall \lambda \in (0,1]$，$\{\tilde{g}_\lambda(n)\}$ 为收敛的区间值函数列，且若令 $\tilde{g}_\lambda^0 = \lim\limits_{n\to\infty} \tilde{g}_\lambda^{(n)}$，则 $\forall x \in X$，有 $\tilde{g}_\lambda(x) = \bigcap\limits_{\lambda < \lambda'} \tilde{g}_{\lambda'}^0(x)$.

**定理 1(单调收敛定理)**　设 $\{\tilde{f}_n\}$ 为 $(Z, \mathscr{F}(Z), \tilde{\mu})$ 中复模糊集值复模糊可积的单调不减序列，$\lim\limits_{n\to\infty} \tilde{f}_n = \tilde{f}$，则 $\tilde{f}$ 也是复模糊集值复模糊可积的，且对于 $A \in \mathscr{F}(Z)$，$\lim\limits_{n\to\infty} \int_A \tilde{f}_n \mathrm{d}\tilde{\mu} = \int_A \tilde{f} \mathrm{d}\tilde{\mu}$.

**证**　由于 $\lim\limits_{n\to\infty} \tilde{f}_n = \tilde{f}$，则 $\lim\limits_{n\to\infty} \mathrm{Re}\tilde{f}_n = \mathrm{Re}\tilde{f}$ 且

$\lim\limits_{n\to\infty}\mathrm{Im}\tilde{f}_n=\mathrm{Im}\tilde{f}$. $\forall\lambda\in(0,1]$,

$$(\tilde{f}_n)_\lambda=(\mathrm{Re}\tilde{f}_n,\mathrm{Im}\tilde{f}_n)_\lambda=(\mathrm{Re}\tilde{f}_{n,\lambda},\mathrm{Im}\tilde{f}_{n,\lambda}),$$

$$\lim_{n\to\infty}(\tilde{f}_n)_\lambda=(\lim_{n\to\infty}\mathrm{Re}\tilde{f}_{n,\lambda},\lim_{n\to\infty}\mathrm{Im}\tilde{f}_{n,\lambda}).$$

由引理 1 可知,

$$\begin{cases}\lim\limits_{n\to\infty}\mathrm{Re}\tilde{f}_{n,\lambda}=f_\lambda^0=[f_\lambda^{0-},f_\lambda^{0+}],\\ \lim\limits_{n\to\infty}\mathrm{Im}\tilde{f}_{n,\lambda}=f_\lambda'=[f_\lambda'^{-},f_\lambda'^{+}],\end{cases}\tag{1}$$

$\forall\{\lambda_n\}\subset(0,1)$,$\lambda_n\downarrow\lambda$,$0<\lambda\leqslant1$. 由引理2 可知,$\forall z\in Z$,

$$\begin{cases}\mathrm{Re}\tilde{f}_{n,\lambda}(z)=\bigcap\limits_{\lambda<\lambda'}f_{\lambda'}^0(z)=\lim\limits_{\lambda_k\to\lambda}f_{\lambda_k}^0(z),\\ \mathrm{Im}\tilde{f}_{n,\lambda}(z)=\bigcap\limits_{\lambda<\lambda'}f_{\lambda'}'(z)=\lim\limits_{\lambda_k\to\lambda}f_{\lambda_k}'(z),\end{cases}\tag{2}$$

故由 $\tilde{f}_n$ 可积得 $\forall\lambda\in(0,1]$,$\lim\limits_{n\to\infty}\mathrm{Re}\tilde{f}_{n,\lambda}$,$\lim\limits_{n\to\infty}\mathrm{Im}\tilde{f}_{n,\lambda}$ 可积,由(1)式知区间值函数 $f_\lambda^0$,$f_\lambda'$ 也可积,结合(2)式知 $\lim\limits_{n\to\infty}\mathrm{Re}\tilde{f}_\lambda$,$\lim\limits_{n\to\infty}\mathrm{Im}\tilde{f}_\lambda$ 均可积,从而 $\tilde{f}$ 可积.

另一方面,由经典 Lebesgue 积分收敛定理知,

$$\begin{cases}\lim\limits_{n\to\infty}(\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R)_\lambda=\lim\limits_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_{n,\lambda}\mathrm{d}(\tilde{\mu}_R)_\lambda=\\ \int_A f_\lambda^0\mathrm{d}(\tilde{\mu}_R)_\lambda,\\ \lim\limits_{n\to\infty}(\int_A\mathrm{Im}\tilde{f}_n\mathrm{d}\tilde{\mu}_I)_\lambda=\lim\limits_{n\to\infty}\int_A\mathrm{Im}\tilde{f}_{n,\lambda}\mathrm{d}(\tilde{\mu}_I)_\lambda=\\ \int_A f_\lambda'\mathrm{d}(\tilde{\mu}_I)_\lambda,\end{cases}$$

此外,$\forall\lambda\in(0,1]$,由模糊集分解定理及广义 Lebesgue 积分定理可得

$$(\lim_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R)_\lambda=\bigcap_{\lambda<\lambda'}\lim_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}(\tilde{\mu}_R)_{\lambda'}=$$

$$\bigcap_{\lambda<\lambda'}\int_A f_{\lambda'}^0\mathrm{d}(\tilde{\mu}_R)_{\lambda'}=\lim_{\lambda_k\to\lambda}\int_A f_{\lambda_k}^0\mathrm{d}(\tilde{\mu}_R)_{\lambda_k}=$$

$$[\int_A(\mathrm{Re}\tilde{f})_{\lambda^-}\mathrm{d}(\tilde{\mu}_R)_{\lambda^-},\int_A(\mathrm{Re}\tilde{f})_{\lambda^+}\mathrm{d}(\tilde{\mu}_R)_{\lambda^+}]=$$

$$(\int_A\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R)_\lambda.$$

因此,$\lim\limits_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R=\int_A\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R$.

同理可证$\lim\limits_{n\to\infty}\int_A\mathrm{Im}\tilde{f}_n\mathrm{d}\tilde{\mu}_I=\int_A\mathrm{Im}\tilde{f}\mathrm{d}\tilde{\mu}_I$,即

$$\lim_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu}=\int_A\tilde{f}\mathrm{d}\tilde{\mu}.$$

**定理 2(法都引理)** 设$\{\tilde{f}_n\}$是$(Z,\mathscr{F}(Z),\tilde{\mu})$上复模糊集值复模糊可积函数序列,则当$\lim\limits_{n\to\infty}\tilde{f}_n$存在时,对于$A\in\mathscr{F}(Z)$,$\int_A\liminf\limits_{n\to\infty}\tilde{f}_n\mathrm{d}\tilde{\mu}\leqslant\liminf\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu}$.

证 令$\tilde{g}_n=\inf\limits_{j\geqslant n}\tilde{f}_j$,则$\tilde{g}_n\leqslant\tilde{f}_j$,$j=n+1,n+2,\cdots$,且$\{\tilde{g}_n\}$单调增加. 故有$\int_A\tilde{g}_n\mathrm{d}\tilde{\mu}\leqslant\int_A\tilde{f}_j\mathrm{d}\tilde{\mu}$;两边关于$j$取下确界,则有$\int_A\tilde{g}_n\mathrm{d}\tilde{\mu}\leqslant\inf\limits_{j\geqslant n}\int_A\tilde{f}_j\mathrm{d}\tilde{\mu}$. 由定理 1 得

$$\int_A\liminf_{n\to\infty}\tilde{f}_n\mathrm{d}\tilde{\mu}=\lim_{n\to\infty}\int_A\tilde{g}_n\mathrm{d}\tilde{\mu}\leqslant\lim_{n\to\infty}\inf_{j\geqslant n}\int_A\tilde{f}_j\mathrm{d}\tilde{\mu}.$$

对偶地有下列定理(证明与定理 2 类似).

**定理 3** 设$\{\tilde{f}_n\}$是$(Z,\mathscr{F}(Z),\tilde{\mu})$上复模糊集值复模糊可积函数序列,则当$\lim\limits_{n\to\infty}\tilde{f}_n$存在时,对于$A\in\mathscr{F}(Z)$,$\int_A\limsup\limits_{n\to\infty}\tilde{f}_n\mathrm{d}\tilde{\mu}\geqslant\limsup\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu}$.

**定理 4(控制收敛定理)** 设$\{\tilde{f}_n\}$是$(Z,\mathscr{F}(Z),\tilde{\mu})$上复模糊集值复模糊可积函数序列,若存在可积的复模糊集值复模糊函数$\tilde{g}$,$\tilde{h}$,$\forall n\in\mathbf{N}$,满足$\tilde{g}\leqslant\tilde{f}_n\leqslant\tilde{h}$,且$\lim\limits_{n\to\infty}\tilde{f}_n=\tilde{f}$,则$\tilde{f}$是复模糊集值复模糊可积的,且对于$A\in\mathscr{F}(Z)$,$\lim\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu}=\int_A\tilde{f}\mathrm{d}\tilde{\mu}$.

证 根据定理 1,显然,$\tilde{f}\in F^*(\mathbf{C})$,由于$\tilde{g}\leqslant\tilde{f}_n\leqslant\tilde{h}(\forall n\in\mathbf{N})$,以及$\forall\lambda\in(0,1]$,对任一固定的$n\in\mathbf{N}$得$\mathrm{Re}\tilde{g}_\lambda\leqslant\mathrm{Re}\tilde{f}_{n,\lambda}\leqslant\mathrm{Re}\tilde{h}_\lambda$,$\mathrm{Im}\tilde{g}_\lambda\leqslant\mathrm{Im}\tilde{f}_{n,\lambda}\leqslant\mathrm{Im}\tilde{h}_\lambda$. 显然$\mathrm{Re}\tilde{g}_\lambda^-\leqslant\mathrm{Re}\tilde{f}_{n,\lambda}^-\leqslant\mathrm{Re}\tilde{h}_\lambda^-$,$\mathrm{Re}\tilde{g}_\lambda^+\leqslant\mathrm{Re}\tilde{f}_{n,\lambda}^+\leqslant\mathrm{Re}\tilde{h}_\lambda^+$. 从而有$|\mathrm{Re}\tilde{f}_{n,\lambda}^-|\leqslant|\mathrm{Re}\tilde{g}_\lambda^-|\vee|\mathrm{Re}\tilde{h}_\lambda^-|$,$|\mathrm{Re}\tilde{f}_{n,\lambda}^+|\leqslant|\mathrm{Re}\tilde{g}_\lambda^+|\vee|\mathrm{Re}\tilde{h}_\lambda^+|$. 由$\tilde{g}$,$\tilde{h}$的可积性知$\mathrm{Re}\tilde{g}_\lambda^-$,$\mathrm{Re}\tilde{g}_\lambda^+$,$\mathrm{Re}\tilde{h}_\lambda^-$,$\mathrm{Re}\tilde{h}_\lambda^+$均 Lebesgue 可积,因而$|\mathrm{Re}\tilde{g}_\lambda^-|\vee|\mathrm{Re}\tilde{h}_\lambda^-|$,$|\mathrm{Re}\tilde{g}_\lambda^+|\vee|\mathrm{Re}\tilde{h}_\lambda^+|$均可积. 由经典 Lebesgu 控制收敛定理可知$\lim\limits_{n\to\infty}\mathrm{Re}\tilde{f}_{n,\lambda}^-$,$\lim\limits_{n\to\infty}\mathrm{Re}\tilde{f}_{n,\lambda}^+$均可积,且有

$$\lim_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_{n,\lambda}\mathrm{d}(\tilde{\mu}_R)_\lambda=\int_A\lim_{n\to\infty}\mathrm{Re}\tilde{f}_n\mathrm{d}(\tilde{\mu}_R)_\lambda.\tag{3}$$

由模糊集分解定理,$\forall z\in Z\subset\mathscr{F}(Z)$,$\mathrm{Re}\tilde{f}_\lambda(z)=\bigcap\limits_{\lambda<\lambda'}\lim\limits_{n\to\infty}\mathrm{Re}\tilde{f}_{n,\lambda}(z)$. 因此,$\mathrm{Re}\tilde{f}_{n,\lambda}$可积,即$\mathrm{Re}\tilde{f}$可积. 同理可得$\mathrm{Im}\tilde{f}$也可积,且

$$\lim_{n\to\infty}\int_A\mathrm{Im}\tilde{f}_{n,\lambda}\mathrm{d}(\tilde{\mu}_R)_\lambda=\int_A\lim_{n\to\infty}\mathrm{Im}\tilde{f}_n\mathrm{d}(\tilde{\mu}_R)_\lambda.\tag{4}$$

综上所述,$\tilde{f}$可积,结合(3)式和(4)式得

$$\lim_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu}=\int_A\tilde{f}\mathrm{d}\tilde{\mu}.$$

**定义 4** 设$(Z,\mathscr{F}(Z),\tilde{\mu})$是复模糊集值模糊测度空间,$\tilde{f}_n$,$\tilde{f}$: $Z\to F_0(\mathbf{C})$为复模糊值模糊可测函数,以及$\tilde{A}\in\mathscr{F}(Z)$,

(i) 称$\{\tilde{f}_n\}$在$\tilde{A}$上几乎处处收敛于$\tilde{f}$,若$\exists\tilde{E}\subset\tilde{A}$,$\tilde{\mu}(\tilde{E})=\tilde{0}$,使得$\{\tilde{f}_n\}$在$\tilde{A}-\tilde{E}$上逐点收敛于$\tilde{f}$,记

在 $\tilde{A}$ 上 $\tilde{f}_n \xrightarrow[A]{a.e} \tilde{f}$.

(ii) 称$\{\tilde{f}_n\}$ 在 $\tilde{A}$ 上几乎一致收敛于 $\tilde{f}$，若 $\exists \tilde{E} \subset \tilde{A}$，$\forall \varepsilon > 0$，$|\tilde{\mu}(\tilde{E})| < \varepsilon$，使得$\{\tilde{f}_n\}$ 在$\tilde{A} - \tilde{E}$ 上逐点一致收敛于 $\tilde{f}$，记在 $\tilde{A}$ 上 $\tilde{f}_n \xrightarrow[A]{a.u} \tilde{f}$；

(iii) 称$\{\tilde{f}_n\}$ 在 $\tilde{A}$ 上伪几乎处处收敛于 $\tilde{f}$，如果 $\exists \tilde{E} \subset \tilde{A}$，$\tilde{\mu}(\tilde{A} - \tilde{E}) = \tilde{\mu}(\tilde{A})$，使得$\{\tilde{f}_n\}$ 在$\tilde{A} - \tilde{E}$ 上逐点收敛于 $\tilde{f}$，记在 $\tilde{A}$ 上 $\tilde{f}_n \xrightarrow[A]{p.a.e} \tilde{f}$；

(iv) 称$\{\tilde{f}_n\}$ 在 $\tilde{A}$ 上伪几乎一致收敛于 $\tilde{f}$，如果 $\exists\{\tilde{E}_k\} \subset \mathscr{F}(Z)$，$(\tilde{\rho})\ \lim\limits_{k\to\infty}\tilde{\mu}(\tilde{A} - \tilde{E}_k) = \tilde{\mu}(\tilde{A})$，使得对于任意固定的 $k = 1,2\cdots$，$\{\tilde{f}_n\}$ 在$\tilde{A} - \tilde{E}_k$ 上逐点一致收敛于 $\tilde{f}$，记在 $\tilde{A}$ 上 $\tilde{f}_n \xrightarrow[A]{p.a.u} \tilde{f}$.

**定理 5**　设$(Z,\mathscr{F}(Z),\tilde{\mu})$ 是复模糊集值模糊测度空间，$\{\tilde{f}_n\}$，$\tilde{f}$ 是$(Z,\mathscr{F}(Z),\tilde{\mu})$ 上的复模糊集值复模糊可测函数，$A \in \mathscr{F}(Z)$，若在 $A$ 上，$\tilde{f}_n \xrightarrow[A]{a.e} \tilde{f}$（几乎处处收敛）. 若 $\tilde{\mu}$ 是零可加的，$\forall \varepsilon_k > 0(k = 1,2)$，其中 $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2$，$\exists n_0$，使得 $\tilde{\mu}(\{x \mid \sup\limits_{n\geqslant n_0}\tilde{f}_n > \int_A \tilde{f}\mathrm{d}\tilde{\mu} + \varepsilon\} \cap A) < \infty + \mathrm{i}\infty$，则

$$\lim_{n\to\infty}\int_A \tilde{f}_n\mathrm{d}\tilde{\mu} = \int_A \tilde{f}\mathrm{d}\tilde{\mu}.$$

**证**　因为在 $A$ 上，$\tilde{f}_n \xrightarrow[A]{a.e} \tilde{f}$，则 $\exists B \in \mathscr{F}(Z)$，$\tilde{\mu}(B) = \tilde{0}$，$\tilde{\mu}$ 零可加，

$$\int_{A\backslash B}\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R = \bigcup_{\lambda\in[0,1]}\lambda\left[\int_{A\backslash B}\mathrm{Re}f_\lambda^-\mathrm{d}\tilde{\mu}_R,\int_{A\backslash B}\mathrm{Re}f_\lambda^+\mathrm{d}\tilde{\mu}_R\right] =$$

$$\bigcup_{\lambda\in[0,1]}\lambda\left[\sup_{\alpha\in[0,\infty)}\alpha\wedge\mathrm{Re}\tilde{\mu}_\lambda^-((A\backslash B)\cap\chi_{F_{\lambda,\alpha,1}}^-),\right.$$

$$\left.\sup_{\alpha\in[0,\infty)}\alpha\wedge\mathrm{Re}\lambda^+((A\backslash B)\cap\chi_{F_{\lambda,\alpha,1}}^+)\right] =$$

$$\bigcup_{\lambda\in[0,1]}\lambda\left[\sup_{\alpha\in[0,\infty)}\alpha\wedge\mathrm{Re}\tilde{\mu}_\lambda^-((A\cap\chi_{F_{\lambda,\alpha,1}}^-)\backslash B),\right.$$

$$\left.\sup_{\alpha\in[0,\infty)}\alpha\wedge\mathrm{Re}\tilde{\mu}_\lambda^+((A\cap\chi_{F_{\lambda,\alpha,1}}^+)\backslash B)\right] =$$

$$\bigcup_{\lambda\in[0,1]}\lambda\left[\sup_{\alpha\in[0,\infty)}\alpha\wedge\mathrm{Re}\tilde{\mu}_\lambda^-(A\cap\chi_{F_{\lambda,\alpha,1}}^-),\right.$$

$$\left.\sup_{\alpha\in[0,\infty)}\alpha\wedge\mathrm{Re}\tilde{\mu}_\lambda^+(A\cap\chi_{F_{\lambda,\alpha,1}}^+)\right] = \int_A\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R.$$

同理可得 $\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R = \int_{A\backslash B}\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R$，$\int_A\mathrm{Im}\tilde{f}\mathrm{d}\tilde{\mu}_I = \int_{A\backslash B}\mathrm{Im}\tilde{f}\mathrm{d}\tilde{\mu}_I$，$\int_A\mathrm{Im}\tilde{f}_n\mathrm{d}\tilde{\mu}_I = \int_{A\backslash B}\mathrm{Im}\tilde{f}_n\mathrm{d}\tilde{\mu}_I$.

因为 $\forall \varepsilon_k > 0(k = 1,2)$，其中 $\varepsilon = \varepsilon_1 + i\varepsilon_2$，$\exists n_0$，使得 $\tilde{\mu}(\{x \mid \sup\limits_{n\geqslant n_0}\tilde{f}_n > \int_A\tilde{f}\mathrm{d}\tilde{\mu} + \varepsilon\} \cap A) < \infty + \mathrm{i}\infty$，$\lim\limits_{n\to\infty}\int_{A\backslash B}\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_{A\backslash B}\tilde{f}\mathrm{d}\tilde{\mu}$. 所以$\lim\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_A\tilde{f}\mathrm{d}\tilde{\mu}$.

**定理 6**　设$(Z,\mathscr{F}(Z),\tilde{\mu})$ 是复模糊集值模糊测度空间，$\{\tilde{f}_n\}$，$\tilde{f}$ 是$(Z,\mathscr{F}(Z),\tilde{\mu})$ 上的复模糊集值复模糊可测函数，$A \in \mathscr{F}(Z)$，若在 $A$ 上$\{\tilde{f}_n\}$ 一致收敛于 $\tilde{f}$，则$\lim\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_A\tilde{f}\mathrm{d}\tilde{\mu}$.

**定理 7**　设$(Z,\mathscr{F}(Z),\tilde{\mu})$ 是复模糊集值模糊测度空间，$\{\tilde{f}_n\}$，$\tilde{f}$ 是$(Z,\mathscr{F}(Z),\tilde{\mu})$ 上的复模糊集值复模糊可测函数，$A \in \mathscr{F}(Z)$，若在 $A$ 上，$\tilde{f}_n \xrightarrow[A]{a.e.u} \tilde{f}$（几乎处处一致收敛），且 $\tilde{\mu}$ 是零可加的，则

$$\lim_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_A\tilde{f}\mathrm{d}\tilde{\mu}.$$

定理 6 和定理 7 的证明过程与定理 5 相仿，此处略.

**定理 8**　设$(Z,\mathscr{F}(Z),\tilde{\mu})$ 是复模糊集值模糊测度空间，$\{\tilde{f}_n\}$，$\tilde{f}$ 是$(Z,\mathscr{F}(Z),\tilde{\mu})$ 上复模糊集值复模糊可测函数，$A \in \mathscr{F}(Z)$，若在 $A$ 上，$\tilde{f}_n \xrightarrow[A]{a.u} \tilde{f}$（几乎一致收敛），$\tilde{\mu}$ 是零可加的，则 $\exists\{B_k\} \subseteq \mathscr{F}(Z)$，$B_1 \supseteq B_2 \supseteq \cdots$，$\tilde{\mu}(B_k) \to 0$，使得

$$\lim_{n\to\infty}\int_{A\backslash B_k}\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_{A\backslash B_k}\tilde{f}\mathrm{d}\tilde{\mu}.$$

**证**　在 $A$ 上，$\tilde{f}_n \xrightarrow[A]{a.u} \tilde{f}$，则 $\exists\{E_k\} \subseteq \mathscr{F}(Z)$，$\tilde{\mu}(E_k) \to 0$，在 $A\backslash E_k$ 上，$\tilde{f}_n \xrightarrow[A]{u} \tilde{f}$（逐点收敛）. 令 $B_k = \bigcap\limits_{i=1}^{k}E_i \subseteq E_k$，则 $A\backslash B_k = \bigcup\limits_{i=1}^{k}(A\backslash E_i)$. $\forall k$，在 $A\backslash E_k$ 上 $\tilde{f}_n \xrightarrow[A]{u} \tilde{f}$，在 $A\backslash B_k$ 上 $\tilde{f}_n \xrightarrow[A]{u} \tilde{f}$，由定理 7 知结论成立.

关于依复模糊集值复模糊测度收敛和伪复模糊集值复模糊测度收敛的相关概念参见文献[17].

**定理 9**　设$(Z,\mathscr{F}(Z),\tilde{\mu})$ 是复模糊集值模糊测度空间，$\{\tilde{f}_n\} \subset F_0(\mathbf{C}^+)$，$\tilde{f} \in F_0(\mathbf{C}^+)$，$\tilde{A} \in \mathscr{F}(Z)$，

(i) $\{\tilde{f}_n\}$ 在 $\tilde{A}$ 上依复模糊集值复模糊测度 $\tilde{\mu}$ 收敛于 $\tilde{f}$，$\tilde{\mu}$ 是自连续的，则$\lim\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_A\tilde{f}\mathrm{d}\tilde{\mu}$；

(ii) $\{\tilde{f}_n\}$ 在 $\tilde{A}$ 上伪依复模糊集值复模糊测度 $\tilde{\mu}$ 收敛于 $\tilde{f}$，$\tilde{\mu}$ 是伪自连续的，则$\lim\limits_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu} = \int_A\tilde{f}\mathrm{d}\tilde{\mu}$.

**证**　(i) 因为$\{\tilde{f}_n\}$ 在 $\tilde{A}$ 上依复模糊集值复模糊测度 $\tilde{\mu}$ 收敛于 $\tilde{f}$，则 $\forall \varepsilon > 0$，$\forall \lambda \in (0,1]$，

$(\tilde{\rho})\ \lim\limits_{n\to\infty}\tilde{\mu}_R(\tilde{A} \cap \{x: |(\mathrm{Re}\tilde{f}_n)_\lambda^- - (\mathrm{Re}\tilde{f})_\lambda^-| \geqslant \varepsilon\}) = \tilde{0}$，

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|\geqslant\varepsilon\})=\bar 0$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_I(\tilde A\cap\{x:|(\mathrm{Im}\tilde f_n)_\lambda^--(\mathrm{Im}\tilde f)_\lambda^-|\geqslant\varepsilon\})=\bar 0$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_I(\tilde A\cap\{x:|(\mathrm{Im}\tilde f_n)_\lambda^+-(\mathrm{Im}\tilde f)_\lambda^+|\geqslant\varepsilon\})=\bar 0$,

其中 $\tilde 0=(\bar 0,\bar 0)$,$\bar 0\in F^*(\mathbf{R})$. $\forall\alpha\in(0,\infty)$,则

$R_{n,\lambda,\alpha+2\varepsilon}^-\subset R_{\lambda,\alpha+\varepsilon}^-\cup\{x:|(\mathrm{Re}\tilde f_n)_\lambda^--(\mathrm{Re}\tilde f)_\lambda^-|\geqslant\varepsilon\}$,

$R_{n,\lambda,\alpha+2\varepsilon}^+\subset R_{\lambda,\alpha+\varepsilon}^+\cup\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|\geqslant\varepsilon\}$.

由 $\tilde\mu$ 的上自连续性得

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R((\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-)\cup(\tilde A\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^--(\mathrm{Re}\tilde f)_\lambda^-|\geqslant\varepsilon\})=\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-)$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R((\tilde A\cap R_{\lambda,\alpha+\varepsilon}^+)\cup(\tilde A\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|\geqslant\varepsilon\})=\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^+)$.

由数列极限定义知,对上述 $\varepsilon>0$,$\exists n\in\mathbf{N}$,使得当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$,

$(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha+2\varepsilon}^-)\leqslant(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-)+\varepsilon$,

$(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha+2\varepsilon}^+)\leqslant(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^+)+\varepsilon$,

当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$,有 $\sup\limits_{\alpha\in(0,\infty)}(\alpha+2\varepsilon)\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha+2\varepsilon}^-)\leqslant\sup\limits_{\alpha\in(0,\infty)}(\alpha+2\varepsilon)\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-)+\varepsilon$,即

$$\sup_{\alpha\in(0,\infty)}\alpha\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha}^-)\leqslant(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha}^-)+\varepsilon.$$

同理可证,当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$ 有

$$\sup_{\alpha\in(0,\infty)}\alpha\wedge(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha}^+)\leqslant(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha}^+)+\varepsilon.$$

从而由复模糊集值复模糊积分定义知

$$\lim_{n\to\infty}\int_A\mathrm{Re}\tilde f_n\mathrm{d}\tilde\mu_R\leqslant\int_A\mathrm{Re}\tilde f\mathrm{d}\tilde\mu_R.$$

另一方面,$\forall\lambda\in(0,1]$,$\forall\alpha\in(0,\infty)$,有

$R_{n,\lambda,\alpha-2\varepsilon}^-\supset R_{\lambda,\alpha-\varepsilon}^-\cup\{x:|(\mathrm{Re}\tilde f_n)_\lambda^--(\mathrm{Re}\tilde f)_\lambda^-|\geqslant\varepsilon\}^C$,

$R_{n,\lambda,\alpha-2\varepsilon}^+\supset R_{\lambda,\alpha-\varepsilon}^+\cup\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|\geqslant\varepsilon\}^C$.

由 $\tilde\mu$ 的下自连续性得

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^-\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^--(\mathrm{Re}\tilde f)_\lambda^-|\geqslant\varepsilon\}^C)=\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^-)$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^+\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|\geqslant\varepsilon\}^C)=\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^+)$.

由数列极限定义知,对上述 $\varepsilon>0$,$\exists n\in\mathbf{N}$,使得当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$,

$(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{n,\lambda,\alpha-2\varepsilon}^-)\geqslant(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^-)-\varepsilon$,

$(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{n,\lambda,\alpha-2\varepsilon}^+)\geqslant(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^+)-\varepsilon$,

当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$,

$\sup\limits_{\alpha\in(0,\infty)}\alpha\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{n,\lambda,\alpha}^-)=\sup\limits_{\alpha\in(2\varepsilon,\infty)}(\alpha-2\varepsilon)\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{n,\lambda,\alpha-2\varepsilon}^-)\geqslant$

$\sup\limits_{\alpha\in(2\varepsilon,\infty)}(\alpha-\varepsilon)\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^-)-\varepsilon=$

$\sup\limits_{\alpha\in(\varepsilon,\infty)}\alpha\wedge(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^-)-\varepsilon.$

同理可证,当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$,

$\sup\limits_{\alpha\in(0,\infty)}\alpha\wedge(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{n,\lambda,\alpha}^+)\geqslant\sup\limits_{\alpha\in(\varepsilon,\infty)}\alpha\wedge(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha-\varepsilon}^+)-\varepsilon$,

从而有 $\lim\limits_{n\to\infty}\int_A\mathrm{Re}\tilde f_n\mathrm{d}\tilde\mu_R\geqslant\int_A\mathrm{Re}\tilde f\mathrm{d}\tilde\mu_R$.

同理可证 $\lim\limits_{n\to\infty}\int_A\mathrm{Im}\tilde f_n\mathrm{d}\tilde\mu_I=\int_A\mathrm{Im}\tilde f\mathrm{d}\tilde\mu_I$.

故 $\lim\limits_{n\to\infty}\int_A\tilde f_n\mathrm{d}\tilde\mu=\int_A\tilde f\mathrm{d}\tilde\mu$.

(ii) 因为 $\{\tilde f_n\}$ 在 $\tilde A$ 上伪依复模糊集值复模糊测度 $\tilde\mu$ 收敛于 $\tilde f$,则 $\forall\varepsilon>0$ 及 $\forall\lambda\in(0,1]$ 有

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^-(x)-(\mathrm{Re}\tilde f)_\lambda^-(x)|<\varepsilon\})=\bar 0$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+(x)-(\mathrm{Re}\tilde f)_\lambda^+(x)|<\varepsilon\})=\bar 0$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_I(\tilde A\cap\{x:|(\mathrm{Im}\tilde f_n)_\lambda^-(x)-(\mathrm{Im}\tilde f)_\lambda^-(x)|<\varepsilon\})=\bar 0$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_I(\tilde A\cap\{x:|(\mathrm{Im}\tilde f_n)_\lambda^+(x)-(\mathrm{Im}\tilde f)_\lambda^+(x)|<\varepsilon\})=\bar 0$.

$\forall\alpha\in(0,\infty)$,则有

$R_{n,\lambda,\alpha+2\varepsilon}^-\subset R_{n,\lambda,\alpha+\varepsilon}^-\cup\{x:|(\mathrm{Re}\tilde f_n)_\lambda^--(\mathrm{Re}\tilde f)_\lambda^-|<\varepsilon\}^C$,$R_{n,\lambda,\alpha+2\varepsilon}^+\subset R_{n_\lambda,\alpha+\varepsilon}^+\cup\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|<\varepsilon\}^C$.

由 $\tilde\mu$ 的上伪自连续性得

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^--(\mathrm{Re}\tilde f)_\lambda^-|<\varepsilon\}^C)=\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-)$,

$(\tilde\rho)\ \lim_{n\to\infty}\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^+\cap\{x:|(\mathrm{Re}\tilde f_n)_\lambda^+-(\mathrm{Re}\tilde f)_\lambda^+|<\varepsilon\}^C)=\tilde\mu_R(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^+)$.

由数列极限定义知,对上述 $\varepsilon>0$,$\exists n\in\mathbf{N}$,使得当 $n\geqslant n_0$ 时,$\forall\lambda\in(0,1]$,

$(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{n,\lambda,\alpha+2\varepsilon}^-)\leqslant(\tilde\mu_R)_\lambda^-(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^-)+\varepsilon$,

$(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{n,\lambda,\alpha+2\varepsilon}^+)\leqslant(\tilde\mu_R)_\lambda^+(\tilde A\cap R_{\lambda,\alpha+\varepsilon}^+)+\varepsilon$,

由上确界性质知，

$$\sup_{\alpha\in(2\varepsilon,\infty)}\alpha\wedge(\tilde{\mu}_R)^-_\lambda(\tilde{A}\cap R^-_{n,\lambda,\alpha})\leqslant\sup_{\alpha\in(\varepsilon,\infty)}\alpha\wedge(\tilde{\mu}_R)^-_\lambda(\tilde{A}\cap R^-_{\lambda,\alpha})+\varepsilon,$$

$$\sup_{\alpha\in(2\varepsilon,\infty)}\alpha\wedge(\tilde{\mu}_R)^+_\lambda(\tilde{A}\cap R^+_{n,\lambda,\alpha})\leqslant\sup_{\alpha\in(\varepsilon,\infty)}\alpha\wedge(\tilde{\mu}_R)^+_\lambda(\tilde{A}\cap R^+_{\lambda,\alpha})+\varepsilon,$$

从而有$\lim_{n\to\infty}\int_A \mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R\leqslant\int_A\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R$.

另一方面，$\forall\lambda\in(0,1]$，$\forall\alpha\in(0,\infty)$，有
$R^-_{n,\lambda,\alpha-2\varepsilon}\supset R^-_{n,\lambda,\alpha-\varepsilon}\cap\{x:|(\mathrm{Re}\tilde{f}_n)^-_\lambda-(\mathrm{Re}\tilde{f})^-_\lambda|<\varepsilon\}$，
$R^+_{n,\lambda,\alpha-2\varepsilon}\supset R^+_{n,\lambda,\alpha-\varepsilon}\cap\{x:|(\mathrm{Re}\tilde{f}_n)^+_\lambda-(\mathrm{Re}\tilde{f})^+_\lambda|<\varepsilon\}$.

由 $\tilde{\mu}$ 的下伪自连续性得

$$(\tilde{\rho})\lim_{n\to\infty}\tilde{\mu}_R(\tilde{A}\cap R^-_{\lambda,\alpha-\varepsilon}\cap\{x:|(\mathrm{Re}\tilde{f}_n)^-_\lambda-(\mathrm{Re}\tilde{f})^-_\lambda|<\varepsilon\})=\tilde{\mu}_R(\tilde{A}\cap R^-_{\lambda,\alpha-\varepsilon}),$$

$$(\tilde{\rho})\lim_{n\to\infty}\tilde{\mu}_R(\tilde{A}\cap R^+_{\lambda,\alpha-\varepsilon}\cap\{x:|(\mathrm{Re}\tilde{f}_n)^+_\lambda-(\mathrm{Re}\tilde{f})^+_\lambda|<\varepsilon\})=\tilde{\mu}_R(\tilde{A}\cap R^+_{\lambda,\alpha-\varepsilon}).$$

由数列极限定义知，对上述 $\varepsilon>0$，$\exists n\in\mathbf{N}$，使得当 $n\geqslant n_0$ 时，$\forall\lambda\in(0,1]$，

$$(\tilde{\mu}_R)^-_\lambda(\tilde{A}\cap R^-_{n,\lambda,\alpha-2\varepsilon})\geqslant(\tilde{\mu}_R)^-_\lambda(\tilde{A}\cap R^-_{\lambda,\alpha-\varepsilon})-\varepsilon,$$

$$(\tilde{\mu}_R)^+_\lambda(\tilde{A}\cap R^+_{n,\lambda,\alpha-2\varepsilon})\geqslant(\tilde{\mu}_R)^+_\lambda(\tilde{A}\cap R^+_{\lambda,\alpha-\varepsilon})-\varepsilon,$$

当 $n\geqslant n_0$ 时，$\forall\lambda\in(0,1]$ 时有

$$\sup_{\alpha\in(0,\infty)}\alpha\wedge(\tilde{\mu}_R)^-_\lambda(\tilde{A}\cap R^-_{n,\lambda,\alpha})\geqslant\sup_{\alpha\in(\varepsilon,\infty)}\alpha\wedge(\tilde{\mu}_R)^-_\lambda(\tilde{A}\cap R^-_{\lambda,\alpha-\varepsilon})-\varepsilon,$$

$$\sup_{\alpha\in(0,\infty)}\alpha\wedge(\tilde{\mu}_R)^+_\lambda(\tilde{A}\cap R^+_{n,\lambda,\alpha})\geqslant\sup_{\alpha\in(\varepsilon,\infty)}\alpha\wedge(\tilde{\mu}_R)^+_\lambda(\tilde{A}\cap R^+_{\lambda,\alpha-\varepsilon})-\varepsilon,$$

所以$\lim_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R\geqslant\int_A\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R$. 故

$$\lim_{n\to\infty}\int_A\mathrm{Re}\tilde{f}_n\mathrm{d}\tilde{\mu}_R=\int_A\mathrm{Re}\tilde{f}\mathrm{d}\tilde{\mu}_R.$$

同理可证$\lim_{n\to\infty}\int_A\mathrm{Im}\tilde{f}_n\mathrm{d}\tilde{\mu}_I=\int_A\mathrm{Im}\tilde{f}\mathrm{d}\tilde{\mu}_I$. 因此

$$\lim_{n\to\infty}\int_A\tilde{f}_n\mathrm{d}\tilde{\mu}=\int_A\tilde{f}\mathrm{d}\tilde{\mu}.$$

## 3　结论

将一般模糊测度概念从经典集、普通模糊集扩展到更为广泛的复模糊集上，研究了模糊复数域上的复模糊集值复模糊测度、复模糊集值复模糊测度空间上的可测函数及其性质，拓广经典测度理论的范围，推广经典测度理论的相应结论；还研究了基于此复模糊集值测度的复模糊集值函数的积分问题，建立复模糊集值复模糊积分基本理论是模糊复分析学的重要工作，这些工作都拓展了模糊测度与模糊积分理论.

## 4　参考文献


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# The Complex Fuzzy Set-Valued Complex Fuzzy Integral and Its Convergence Theorem

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**Abstract:** The concepts of complex fuzzy set-valued complex fuzzy measure and the complex fuzzy set-valued measurable function,the concepts of complex fuzzy set-valued complex fuzzy integral and its basic properties are introduced. And then the convergence problem of complex fuzzy set-valued complex fuzzy integral are studied,it' s some important convergence theorems are obtained,such as the monotone convergence theorem,the Fatou theorem,the control convergence theorem.

**Key words:** complex fuzzy set-valued measure; complex fuzzy set-valued measurable function; complex fuzzy set-valued complex fuzzy integral; convergence theorem

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