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一类(φ_1,φ_2)-Laplace 差分系统周期解的存在性

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摘要: 利用极小化原理研究了一类(φ_1,φ_2)-Laplace 差分系统周期解的存在性问题. 借助N-函数的概念及其性质,在位势函数满足次凸性条件、次(p,q)次线性增长条件和次p次线性增长条件下,获得了系统周期解的一些存在性准则.

关键词: 差分系统;(φ_1,φ_2)-Laplace;极小化原理;周期解

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0 引言

自2003年郭志明等[1-2]将临界点理论应用到自共轭差分方程周期解与次调和解的存在性问题的研究中以来,变分法已成为研究差分系统各种解的存在性与多重性问题的重要工具[3-14]. 特别地,J. Mawhin[10-11]研究了如下一般的非线性差分系统周期解的存在性与多重性问题:

$$\Delta\varphi[\Delta u(n-1)] = \nabla_u F[n,u(n)] + h(n)\ (n\in\mathbf{Z}), \tag{1}$$

其中φ是$X\subset\mathbf{R}^N$到$Y\subset\mathbf{R}^N$的同胚映射. 通过对X和Y进行适当限制,J. Mawhin将φ分为经典同胚映射、有界同胚映射和奇异同胚映射,进而利用变分法,在F满足不同的增长条件下获得了系统(1)周期解的一系列存在性与多重性结果. 在文献[10-11]的启发下,Wang Yun等[4]考虑了如下非线性差分系统周期解的多重性问题:

$$\begin{cases}\Delta\varphi_1(\Delta u_1(t-1)) = \nabla_{u_1}F(t,u_1(t),u_2(t)) + h_1(t),\\ \Delta\varphi_2(\Delta u_2(t-1)) = \nabla_{u_2}F(t,u_1(t),u_2(t)) + h_2(t),\end{cases} \tag{2}$$

其中Δ是向前差分算子,φ_m和F满足如下条件:

(φ_0) $\varphi_m:\mathbf{R}^N\to B_a\subset\mathbf{R}^N(a\in(0,+\infty])(N\geqslant 1)$是一个同胚映射,$\varphi_m(0)=0$,$\varphi_m=\nabla\Phi_m$,其中$\Phi_m\in C^1(\mathbf{R}^N,[0,+\infty))$严格凸且

$$\Phi_m(0)=0,m=1,2;$$

(F_1) $F:Z\times\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R},(t,x_1,x_2)\to F(t,x_1,x_2)$对所有的$(x_1,x_2)\in\mathbf{R}^N\times\mathbf{R}^N$关于$t$是$T$-周期($T>1$是一个正整数),且对每个$t\in Z[1,T]$,关于$(x_1,x_2)$连续可微,其中

$$Z[a,b]=\{a,a+1,a+2,\cdots,b\},a,b\in\mathbf{N},$$
$$x_1=(x_1^{(1)},x_2^{(1)},\cdots,x_N^{(1)})^{\mathrm{T}},$$
$$x_2=(x_1^{(2)},x_2^{(2)},\cdots,x_N^{(2)})^{\mathrm{T}};$$

(H) $h_i:Z[1,T]\to\mathbf{R}$满足

$$\sum_{t=1}^{T}h_i(t)=0(i=1,2).$$

通过利用文献[15]中的定理4.12和文献[16]中的广义鞍点定理,当位势函数关于(x_1,x_2)部分周期时,文献[4]分别在经典同胚和有界同胚情形下建立了系统(2)周期解的多重性准则. 文献[5]和文献[14]也利用变分法分别研究了类似于(2)式的差分系统同宿轨道和周期解的存在性问题,获得了一系列有趣的存在性准则.

注1 条件(φ_0)首先由文献[10]引进,它被用来刻画如下2种同胚:

(i) 若$a=+\infty$,称$\varphi_m(m=1,2)$为经典同胚;

(ii) 若$a<+\infty$,称$\varphi_m(m=1,2)$为有界同胚.

受文献[4-5,14,17]的启发,本文将利用变分法针对经典同胚的情形来研究系统(2)周期解的存在性问题,将在位势函数具有凸性增长、次(p,q)次线性增长和次p次线性增长的条件下,给出系统(2)周期解的一些新的存在性准则.

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1 相关引理

令 $E_T=\{v:=\{v(t)\}\mid v(t+T)=v(t),v(t)\in\mathbf{R}^N,t\in\mathbf{Z}\}$. 易知 E_T 的维数是 NT. 对于 $v\in E_T$,令

$$\|v\|_\infty=\max_{t\in Z[1,T]}|v(t)|.$$

对于 $1<s<+\infty$ 和 $v\in E_T$,定义

$$\|v\|_s=\Big(\sum_{t=1}^T|\Delta v(t)|^s+\sum_{t=1}^T|v(t)^s|\Big)^{1/s}.$$

令 $E=E_T\times E_T$. 对给定 $1<p,q<+\infty$,在 E 上定义范数 $\|u\|=\|u_1\|_p+\|u_2\|_q$,其中 $u=(u_1,u_2)^{\mathrm T}\in E$. 令

$$W=\{u=(u_1,u_2)^{\mathrm T}\in E\,|\,u_m(1)=\cdots=u_m(T)=\frac{1}{T}\sum_{t=1}^T u_m(t),m=1,2\},$$

$$Y=\{(u_1,u_2)^{\mathrm T}\in E\,\Big|\,\sum_{t=1}^T u_m(t)=0,m=1,2\},$$

则 $E=W\oplus Y$ 且 $\forall u\in E$,有 $u=\bar u+\tilde u$,其中

$$\bar u=(\bar u_1,\bar u_2)^{\mathrm T}\in W,\tilde u=(\tilde u_1,\tilde u_2)^{\mathrm T}\in Y,$$

$$\bar u_m=\frac{1}{T}\sum_{t=1}^T u_m(t),u_m=\bar u_m+\tilde u_m,m=1,2.$$

此外,易知 $\Delta\tilde u_m=\Delta u_m,m=1,2$. 上述定义和相关表述可参见文献[4-5,14].

定义 1[18] 令 ψ: $[0,+\infty)\to[0,+\infty)$ 为右连续单调递增函数且满足

(i) $\psi(0)=0$;

(ii) $\lim\limits_{t\to+\infty}\psi(t)\to+\infty$;

(iii) 当 $t>0$ 时,$\psi(t)>0$,

称定义在 R 上的函数 $\Psi(t)=\int_0^{|t|}\psi(s)\,\mathrm ds$ 为 N-函数.

由 N-函数的定义易知 $\Psi(0)=0$ 且 Ψ 是严格凸的,若

$$\sup_{t>0}\Psi(2t)/\Psi(t)<+\infty,$$

则称 N-函数 Ψ 满足全局 Δ_2 条件.

定义 2[18] 定义 $\tilde\Psi$: $[0,+\infty)\to\mathbf R$ 为

$$\tilde\Psi(t)=\max_{s\geqslant0}\{ts-\Psi(s)\},$$

称 $\tilde\Psi$ 为 N-函数的补. $\tilde\Psi$ 也是 N-函数且有 $\tilde{\tilde\Psi}=\Psi$. 对于 Ψ 和 $\tilde\Psi$,有如下形式的 Young 不等式[18-19]:

$$st\leqslant\Psi(s)+\tilde\Psi(t),s,t\geqslant0,\tag{3}$$

且由文献[20]中的 Lemma A.2 可得

$$\tilde\Psi(\psi(t))\leqslant\Psi(2t),t\geqslant0.\tag{4}$$

定义 3[21] 设 $G:\mathbf R^N\times\mathbf R^N\to\mathbf R$,若存在正整数 λ,μ 使得

$$G(\lambda(x_1+x_2),\lambda(y_1+y_2))\leqslant\mu(G(x_1,y_1)+G(x_2,y_2)),$$

则称函数 G 是(λ,μ)-次凸的.

定义 4[15] 设 E 是一个实 Banach 空间,$\varphi\in C^1(E,\mathbf R)$,若 $\forall\{u_n\}\subset E$,$\varphi(u_n)$ 有界且 $\varphi'(u_n)\to0(n\to\infty)$,$\{u_n\}$ 必有收敛的子列,则称泛函 φ 在 E 上满足 Palais-Smale 条件(简称(PS)条件).

引理 1[22] (极小化原理) 若泛函 $\varphi\in C^1(E,\mathbf R)$ 有下(上)界且满足(PS)条件,则

$$c=\inf_{u\in E}\varphi(u)\ (c=\sup_{u\in E}\varphi(u))$$

是 φ 的一个临界值.

引理 2[4] $\forall u=(u_1,u_2)^{\mathrm T},v=(v_1,v_2)^{\mathrm T}\in E$,下面的等式成立:

$$-\sum_{t=1}^T(\Delta\varphi_1(\Delta u_1(t-1)),v_1(t))=\sum_{t=1}^T(\varphi_1(\Delta u_1(t)),\Delta v_1(t)),-\sum_{t=1}^T(\Delta\varphi_2(\Delta u_2(t-1)),v_2(t))=\sum_{t=1}^T(\varphi_2(\Delta u_2(t)),\Delta v_2(t)).$$

引理 3[4] 令 $L:Z[1,T]\times\mathbf R^N\times\mathbf R^N\times\mathbf R^N\times\mathbf R^N\to\mathbf R$,$(t,x_1,x_2,y_1,y_2)\to L(t,x_1,x_2,y_1,y_2)$ 且假设 $\forall t\in Z[1,T]$,L 关于(x_1,x_2,y_1,y_2)是连续可微的. 定义

$$J(u)=J(u_1,u_2)=\sum_{t=1}^T L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),$$

则能量泛函 $J:E\to\mathbf R$ 在 E 上连续可微且 $\forall u,v\in E$,有

$$\langle J'(u),v\rangle=\langle J'(u_1,u_2),(v_1,v_2)\rangle=\sum_{t=1}^T[(D_{x_1}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),v_1(t))+(D_{y_1}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),\Delta v_1(t))+(D_{x_2}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),v_2(t))+(D_{y_2}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),\Delta v_2(t))],$$

令

$$L(t,x_1,x_2,y_1,y_2)=\Phi_1(y_1)+\Phi_2(y_2)+F(t,x_1,x_2)+(h_1(t),x_1)+(h_2(t),x_2),$$

则

$$J(u)=J(u_1,u_2)=\sum_{t=1}^T[\Phi_1(\Delta u_1(t))+\Phi_2(\Delta u_2(t))+F(t,u_1(t),u_2(t))+(h_1(t),u_1(t))+(h_2(t),u_2(t))].$$

由(φ_0),$(\mathrm F_1)$,引理 2 和引理 3 得

$$\langle J'(u),v\rangle=\langle J'(u_1,u_2),(v_1,v_2)\rangle=\sum_{t=1}^T[(\varphi_1(\Delta u_1(t)),\Delta v_1(t))+(\varphi_2(\Delta u_2(t)),\Delta v_2(t))+(\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t))+(\nabla_{u_2}F(t,u_1(t),$$

$u_2(t)),v_2(t))+(h(t),u_1(t))+(h_2(t),u_2(t))]=$
$-\sum_{t=1}^{T}[(\Delta\varphi_1(\Delta u_1(t-1)),v_1(t))+(\Delta\varphi_2(\Delta u_2(t-1)),v_2(t))-(\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t))-(\nabla_{u_2}F(t,u_1(t),u_2(t)),v_2(t))-(h_1(t),u_1(t))-(h_2(t),u_2(t))]$,

则显然 J 在 E 中的临界点是系统(2)的 T-周期解.

引理 4[7]　设 $u=(u_1,u_2)^{\mathrm T}\in Y,s>1,s'>1$ 且 $1/s+1/s'=1$,则

$$\max_{t\in Z[1,T]}|u_m(t)|\leqslant C(s')\left(\sum_{t=1}^{T}|\Delta u_m(t)|^s\right)^{1/s},m=1,2,$$

$$\sum_{t=1}^{T}|u_m(t)|^s\leqslant C(s,s')\sum_{t=1}^{T}|\Delta u_m(t)|^s,m=1,2,$$

其中

$$C(s')=\min\left\{\frac{(T-1)^{s+1/s}}{T},\left(\frac{(T+1)^{s+1}-2}{T^s(s+1)}\right)^{1/s}\right\},$$

$$C(s,s')=\min\left\{\frac{(T-1)^{2s-1}}{T^{s-1}},\frac{T^{s-1}\Theta(s,s')}{(s'+1)^{s/s'}}\right\},$$

$$\Theta(s,s')=\sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)^{s'+1}+\left(1-\frac{t}{T}+\frac{1}{T}\right)^{s'+1}-\frac{2}{T^{s'+1}}\right]^{s/s'}.$$

引理 5[14]　$\forall u=(u_1,u_2)^{\mathrm T}\in E$,若 $\|u\|\to\infty$,则

$$|\bar{u}_m|+\sum_{t=1}^{T}|\Delta u_m(t)|^s\to\infty,$$

其中 $m=1$(或 2),$s=p$(或 q).

2　定理及其证明

令 $p>1,p'>1,q>1,q'>1$ 且满足 $1/p+1/p'=1,1/q+1/q'=1$. 假设

$$F(t,x_1,x_2)=G(t,x_1,x_2)+H(t,x_1,x_2),$$

且满足

(F_2)　$\forall t\in Z[1,T],G(t,\cdot,\cdot)$ 是(λ,μ)-次凸的,其中 $\lambda,\mu>1/2$ 且 $\mu<2^{r-1}\lambda^r,r=\min\{p,q\}$;

(F_3)　存在常数 $C_m>0,D_1>0,k_{m_1}>0,k_{m_2}>0,\alpha_1\in[0,p-1],\alpha_2\in[0,q-1]$ 和满足如下性质的 2 个非负函数 $\omega_m\in C([0,+\infty),\mathbf R^+\cup\{0\})$,

(i) $\omega_m(0)=0,\omega_m(t)\to+\infty$(当 $t\to\infty$ 时),$\omega_m(t)>0$(当 $t>0$ 时);

(ii) $\omega_m(s)\leqslant\omega_m(t),0\leqslant s\leqslant t,s,t\in[0,\infty)$;

(iii) $\omega_m(s+t)\leqslant C_m(\omega_m(s)+\omega_m(t)),\forall s,t\in[0,\infty)$;

(iv) $\omega_1(t)\leqslant k_{11}|t|^{\alpha_1}+k_{12},\omega_2(t)\leqslant k_{21}|t|^{\alpha_2}+k_{22},\forall s,t\in[0,+\infty)$;

(v) $\sup_{t>0}W_m(2t)/W_m(t)<+\infty$,且 $W_1(s+t)\leqslant D_1(W_1(s)+W_1(t)),\forall s,t\in[0,+\infty)$,其中 $W_m(t)=\int_0^t\omega_m(s)\mathrm ds,m=1,2$,以及函数 f_i,g_i,l_i: $Z[1,T]\to\mathbf R^+,i=1,2$,使得 $\forall(x_1,x_2)\in\mathbf R^N\times\mathbf R^N$ 和 $t\in Z[1,T]$,有

$$|\nabla_{x_1}H(t,x_1,x_2)|\leqslant f_1(t)\omega_1(|x_1|)+g_1(t)\tilde W_1^{-1}(W_2(|x_2|))+l_1(t),$$

$$|\nabla_{x_2}H(t,x_1,x_2)|\leqslant f_2(t)\omega_2(|x_2|)+g_2(t)\tilde W_2^{-1}(W_1(|x_1|))+l_2(t).$$

注 2　由(F_3)中的条件(v)可知存在一个正整数 k_{0m},使得 $\forall t>0$,有

$$W_m(2t)\leqslant k_{0m}W_m(t),m=1,2.\tag{5}$$

为方便起见,以下记

$$M_1=\sum_{t=1}^{T}f_1(t),M_2=\sum_{t=1}^{T}g_1(t),M_3=\sum_{t=1}^{T}l_1(t),$$

$$M_4=\sum_{t=1}^{T}f_2(t),M_5=\sum_{t=1}^{T}g_2(t),M_6=\sum_{t=1}^{T}l_2(t),$$

$$M_7=\sum_{t=1}^{T}|h_1(t)|,M_8=\sum_{t=1}^{T}|h_2(t)|,$$

$$L_1=\frac{2M_1k_{11}C_1}{\alpha_1+1}C^{\alpha_1+1}(p')+\frac{M_2k_{11}}{\alpha_1+1}C^{\alpha_1+1}(p')+\frac{M_5D_1k_{11}}{\alpha_1+1}C^{\alpha_1+1}(p'),$$

$$L_2=M_3C(p')+2C_1M_1k_{12}C(p')+M_2k_{12}C(p')+M_5D_1k_{12}(p'),$$

$$L_3=\frac{2C_2M_4k_{21}}{\alpha_2+1}C^{\alpha_2+1}(q')+\frac{M_5k_{21}}{\alpha_2+1}C^{\alpha_2+1}(q'),$$

$$L_4=C_2M_4k_{22}C(q')+k_{22}C_2M_4C(q')+M_5k_{22}C(q')+M_6C(q').$$

引理 6　假设$(\varphi_0)(a=+\infty)$,$(F_1)\sim(F_3)$和下列条件成立:

(Φ_0) 存在正常数 d_1,d_2,使得

$\Phi_1(x_1)+\Phi_2(x_2)\geqslant d_1|x_1|^p+d_2|x_2|^q,\forall x_1,x_2\in\mathbf R^N$,

则

$$J(u)\geqslant d_1\sum_{t=1}^{T}|\Delta u_1(t)|^p+d_2\sum_{t=1}^{T}|\Delta u_2(t)|^q-L_1\left(\sum_{t=1}^{T}|\Delta u_1(t)|^q\right)^{(\alpha_2+1)/q}-(L_2+C_8(q'))\cdot\left(\sum_{t=1}^{T}|\Delta u_2(t)|^p\right)^{1/p}-L_3\left(\sum_{t=1}^{T}|\Delta u_2(t)|^q\right)^{(\alpha_2+1)/q}-(L_4+C_9(q'))\left(\sum_{t=1}^{T}|\Delta u_2(t)|^q\right)^{1/q}-2^{\beta/2+1}A_0\mu\Big(C^{\beta}(p')\cdot\left(\sum_{t=1}^{T}|\Delta u_1(t)|^p\right)^{\beta/p}+C^{\beta}(q')\left(\sum_{t=1}^{T}|\Delta u_2(t)|^q\right)^{\beta/q}+$$

$$(W_1(|\bar{u}_1|)+W_2(|\bar{u}_2|))\Big[\frac{1}{W_1(|\bar{u}_1|)+W_2(|\bar{u}_2|)}\cdot\Big(\frac{1}{\mu}\sum_{t=1}^{T}G(t,\lambda\bar{u}_1,\lambda\bar{u}_2)+\sum_{t=1}^{T}H(t,\bar{u}_1,\bar{u}_2)\Big)-\max\{C_1k_{01}M_1+M_5D_1,M_2+C_2M_4k_{02}\}\Big]-\max\{M_7,M_8\}(|\bar{u}_1|+|\bar{u}_2|)-A_0T,\tag{6}$$

其中 $A_0=\max\limits_{\substack{t\in Z[1,T]\\ |x_1|\leqslant 1,|x_2|\leqslant 1}}G(t,x_1,x_2)$.

证 类似于文献[7]中的讨论可知,若(F_2)成立,则 $\forall t\in Z[1,T]$ 和$(x_1,x_2)\in\mathbf{R}^N\times\mathbf{R}^N$,有

$$G(t,x_1,x_2)\leqslant(2^{\beta/2+1}\mu(|x_1|^{\beta}+|x_2|^{\beta})+1)A_0,$$

其中 $\beta=\log_{2\lambda}(2\mu)$ 且 $0<\beta<r$. 从而

$$\sum_{t=1}^{T}G(t,-\tilde{u}_1(t),-\tilde{u}_2(t))\leqslant\sum_{t=1}^{T}(2^{\beta/2+1}\mu(|-\tilde{u}_1(t)|^{\beta}+|-\tilde{u}_2(t)|^{\beta})+1)A_0\leqslant A_0T+2^{\beta/2+1}A_0\mu\Big(C^{\beta}(p')\cdot\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{\beta/p}+C^{\beta}(q')\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{\beta/q}\Big).\tag{7}$$

由(F_3),引理 4 和(3)~(5)式可得

$$\sum_{t=1}^{T}|H(t,u_1(t),\bar{u}_2)-H(t,\bar{u}_1,\bar{u}_2)|=\sum_{t=1}^{T}\Big|\int_0^1(\nabla_{x_1}H(t,\bar{u}_1+s\tilde{u}_1(t),\bar{u}_2),\tilde{u}_1(t))\mathrm{d}s\Big|\leqslant C_1k_{01}M_1W_1(|\bar{u}_1|)+\frac{2M_1k_{11}C_1}{\alpha_1+1}C^{\alpha_1+1}(p')\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{(\alpha_1+1)/p}+M_2W_2(|\bar{u}_2|)+2C_1M_1k_{12}C(p')\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{1/p}+M_3C(p')\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{1/p}+\frac{M_2k_{11}}{\alpha_1+1}C^{\alpha_1+1}(p')\cdot\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{(\alpha_1+1)/p}+M_2k_{12}C(p')\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{1/p},\tag{8}$$

$$\sum_{t=1}^{T}|H(t,u_1(t),u_2(t))-H(t,u_1(t),\bar{u}_2)|=\sum_{t=1}^{T}\Big|\int_0^1(\nabla_{x_2}H(t,u_1(t),\bar{u}_2+s\tilde{u}_2(t)),\tilde{u}_2(t))\mathrm{d}s\Big|\leqslant C_2M_4k_{02}W_2(|\bar{u}_2|)+\frac{(2C_2M_4+M_5)k_{21}}{\alpha_2+1}C^{\alpha_2+1}(q')\cdot\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{(\alpha_2+1)/q}+(C_2M_4+M_5)k_{22}C(q')\cdot\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{1/q}+k_{22}C_2M_4C(q')\cdot\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{1/q}+M_5D_1W_1(|\bar{u}_1|)+M_6C(q')\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{1/q}+\frac{M_5C_1k_{11}}{\alpha_1+1}C^{\alpha_1+1}(p')\cdot\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{(\alpha_1+1)/p}+M_5D_1k_{12}C(p')\cdot\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{1/p}.\tag{9}$$

又由引理 4 可得

$$\Big|\sum_{t=1}^{T}[(h_1(t),\bar{u}_1+\tilde{u}_1(t))+(h_2(t),\bar{u}_2+\tilde{u}_2(t))]\Big|\leqslant|\bar{u}_1|\sum_{t=1}^{T}|h_1(t)|+\|\tilde{u}_1\|_{\infty}\sum_{t=1}^{T}|h_1(t)|+|\bar{u}_2|\sum_{t=1}^{T}|h_2(t)|+\|\tilde{u}_2\|_{\infty}\sum_{t=1}^{T}|h_2(t)|\leqslant M_7|\bar{u}_1|+M_7C(p')\cdot\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{1/p}+M_8|\bar{u}_2|+M_8C(q')\cdot\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{1/q}.\tag{10}$$

因此,由(Φ_0)和(7)~(10)式可得

$$J(u)=J(u_1,u_2)=\sum_{t=1}^{T}[\Phi_1(\Delta u_1(t))+\Phi_2(\Delta u_2(t))+F(t,u_1(t),u_2(t))+(h_1(t),u_1(t))+(h_2(t),u_2(t))]=\sum_{t=1}^{T}[\Phi_1(\Delta u_1(t))+\Phi_2(\Delta u_2(t))+G(t,u_1(t),u_2(t))+H(t,u_1(t),u_2(t))+(h_1(t),u_1(t))+(h_2(t),u_2(t))]\geqslant d_1\sum_{t=1}^{T}|\Delta u_1(t)|^{p}+d_2\sum_{t=1}^{T}|\Delta u_2(t)|^{q}-L_1\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{(\alpha_1+1)/p}-(L_2+M_7C(q'))\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{1/p}-L_3\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{(\alpha_2+1)/q}-(L_4+M_8C(q'))\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{1/q}-2^{\beta/2+1}A_0\mu\Big(C^{\beta}(p')\cdot\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^{p}\Big)^{\beta/p}+C^{\beta}(q')\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^{q}\Big)^{\beta/q}\Big)+(W_1(|\bar{u}_1|)+W_2(|\bar{u}_2|))\Big[\frac{1}{W_1(|\bar{u}_1|)+W_2(|\bar{u}_2|)}\cdot\Big(\frac{1}{\mu}\sum_{t=1}^{T}G(t,\lambda\bar{u}_1,\lambda\bar{u}_2)+\sum_{t=1}^{T}H(t,\bar{u}_1,\bar{u}_2)\Big)-\max\{C_1M_1k_{01}+M_5D_1,M_2+C_2M_4k_{02}\}\Big]-\max\{M_7,M_8\}(|\bar{u}_1|+|\bar{u}_2|)-A_0T.$$

注3 受篇幅所限,引理 6 的推理过程有所省略.

定理 1 假设$(\varphi_0)(a=+\infty)$,(F_1)~(F_3)(其中 $\alpha_1\in[0,p-1)$ 且 $\alpha_2\in[0,q-1)$)和下列条件成立:

(W) 对于$(x_1,x_2)\in\mathbf{R}^N\times\mathbf{R}^N$,

$$\lim_{|x_1|+|x_2|\to\infty}\frac{W_1(|x_1|)+W_2(|x_2|)}{|x_1|+|x_2|}=+\infty;$$

(F_4) 对于$(x_1,x_2)\in\mathbf{R}^N\times\mathbf{R}^N$,

$$\lim_{|x_1|+|x_2|\to\infty}\frac{1}{W_1(|x_1|)+W_2(|x_2|)}\Big(\frac{1}{\mu}\sum_{t=1}^{T}G(t,\lambda x_1,\lambda x_2)+\sum_{t=1}^{T}H(t,x_1,x_2)\Big)\geqslant K_1,$$

其中 $K_1=\max\{C_1M_1k_{01}+M_5D_1,M_2+C_2M_4k_{02}\}$，则系统(2)在 E 中至少有 1 个 T-周期解.

证　由引理 5 知，若 $\|(u_1,u_2)\|\to\infty$，则 $|\bar{u}_m|+\sum_{t=1}^{T}|\Delta u_m(t)|^s\to\infty$，其中 $m=1$(或 2)且 $s=p$(或 q). 又注意到 $\alpha_1\in[0,p-1)$ 且 $\alpha_2\in[0,q-1)$ 以及 $0<\beta<r<\min\{p,q\}$. 由(F_4)，(W)和引理 6 易得，当 $\|(u_1,u_2)\|\to\infty$ 时，有

$$J(u)=J(u_1,u_2)\to\infty. \qquad (11)$$

(11)式表明 J 有下界. 假设序列$\{u_n=(u_{1n},u_{2n})\}\subset E$ 满足 $J(u_n)$ 有界且 $J'(u_n)\to 0$，由(11)式可知序列$\{u_n\}$有界. 注意到 E 的维数有限，则$\{u_n\}$必有收敛的子列，从而(PS)条件成立. 由引理 1 易知，J 在 E 中至少有 1 个临界点 u^* 使得

$$J(u^*)=c=\inf_{u\in E}(u).$$

u^* 即为系统(2)的 T-周期解.

注 4　定理 1 不同于文献[4]中的结果. 特别地，在文献[4]中，$\nabla_{x_1}H(t,x_1,x_2)$ 只能被 $|x_1|$ 的函数控制，同时 $\nabla_{x_2}H(t,x_1,x_2)$ 只能被 $|x_2|$ 的函数控制，而在条件(F_3)中，$\nabla_{x_1}H(t,x_1,x_2)$ 和 $\nabla_{x_2}H(t,x_1,x_2)$ 可以被$(|x_1|,|x_2|)$的函数控制. 所以条件(F_3)比文献[4]中的相应条件更弱一些，存在满足条件(F_3)而不满足文献[4]中的相应条件的系统. 例如，令 $p=2,q=3,\varphi_1(x)=2|x|+3|x|^2,\varphi_2(x)=3|x|^3+4|x|^3,G(t,x_1,x_2)=a_1(t)|x_1|^{3/2}+a_2(t)|x_2|^{11/6}$，$H(t,x_1,x_2)=b_1(t)|x_1|^{4/3}+b_2(t)|x_2|^{5/3}+\ln(1+|x_1|^2)\ln(1+|x_2|^2)$，其中 $a_i,b_i:Z\to\mathbf{R}^+$ 是 T-周期的，T 为大于 1 的整数. 这个例子同时也满足定理 1 的其他条件.

定理 2　假设 $p=q$，$(\varphi_0)(a=+\infty)$，(F_1)，(F_3)(其中 $\alpha_1=p-1$ 且 $\alpha_2=p-1$)，(W)，(F_4)和下列条件成立:

$(F_2)'$　$\forall t\in Z[1,T]$，$G(t,\cdot,\cdot)$是(λ,μ)-次凸的，其中 $\lambda>1/2$ 且 $\mu=2^{r-1}\lambda^r,r=p$.

$(\Phi_0)'$　存在常数

$$d_1>L_1+2^{p/2+1}A_02^{p-1}\lambda^pC^p(p'),$$

$$d_2>L_3+2^{p/2+1}A_02^{p-1}\lambda^pC^p(p'),$$

使得

$$\Phi_1(x_1)+\Phi_2(x_2)\geqslant d_1|x_1|^p+d_2|x_2|^q,\forall x_1,x_2\in\mathbf{R}^N,$$

则系统(2)在 E 中至少有 1 个 T-周期解.

证　类似于文献[17]和定理 1 中的讨论可知，若$(F_2)'$成立，则(7)式成立，且 $\beta=\log_{2\lambda}(2\mu)=r=p$. 当 $\alpha_1=p-1,\alpha_2=p-1,\beta=p$ 时，由引理 6 可得

$$\begin{aligned}J(u)\geqslant&(d_1-L_1-2^{p/2+1}A_0\mu C^p(p'))\sum_{t=1}^{T}|\Delta u_1(t)|^p+(d_2-L_3-2^{p/2+1}A_0\mu C^p(p'))\sum_{t=1}^{T}|\Delta u_2(t)|^q-(L_2+\\&M_7C(p'))\Big(\sum_{t=1}^{T}|\Delta u_1(t)|^p\Big)^{1/p}-(L_4+M_8C(q'))\cdot\Big(\sum_{t=1}^{T}|\Delta u_2(t)|^q\Big)^{1/q}+(W_1(|\bar{u}_1|)+W_2(|\bar{u}_2|))\cdot\\&\Big[\frac{1}{W_1(|\bar{u}_1|)+W_2(|\bar{u}_2|)}\Big(\frac{1}{\mu}\sum_{t=1}^{T}G(t,\lambda\bar{u}_1,\lambda\bar{u}_2)+\sum_{t=1}^{T}H(t,\bar{u}_1,\bar{u}_2)\Big)-\max\{C_1M_1k_{01}+M_5D_1,M_2+\\&C_2M_4k_{02}\}\Big]-\max\{M_7,M_8\}(|\bar{u}_1|+|\bar{u}_2|)-A_0T.\end{aligned}$$

注意到 $\mu=2^{p-1}\lambda^p,d_1>L_1+2^{p/2+1}A_0\mu C^p(p'),d_2>L_3+2^{p/2+1}A_0\mu C^p(p')$，则类似于定理 1 的讨论进行证明即可.

注 5　在定理 1 的基础上，定理 2 进一步考虑了当 $p=q$ 时系统(2)的情形，在这种情形下，$G(t,x_1,x_2)$是(λ,μ)-次凸的，其中 μ 可以取到 $2^{r-1}\lambda^r,r=p$，且 $H(t,x_1,x_2)$的限制可以放宽至 $\alpha_1=\alpha_2=p-1$ 的情形. 所以定理 2 并不同于定理 1. 例如，令 $\varphi_1(x)=3A_1|x|^2+4|x|^3,\varphi_2(x)=3A_2|x|^2+4|x|^3,G(t,x_1,x_2)=a_1(t)|x_1|^3+a_2(t)|x_2|^3,H(t,x_1,x_2)=b_1(t)|x_1|^3+b_2(t)|x_2|^3+\ln(1+|x_1|^2)\ln(1+|x_2|^2)$，其中

$$A_1>\Big(4\sum_{t=1}^{T}b_1(t)+3T+2^{9/2}\max_{t\in Z[1,T]}(a_1(t)+a_2(t))\Big)(C(3/2))^3,$$

$$A_2>3\Big(4\sum_{t=1}^{T}b_2(t)+T\Big)(C(2/3))^3/p,$$

这里 $a_i,b_i:Z\to\mathbf{R}^+,i=1,2$ 是 T-周期的，T 为大于 1 的整数，且 $a_i(i=1,2)$满足

$$\sum_{t=1}^{T}(a_1(t)+a_2(t))>\max\{C_1k_{01}M_1+M_5D_1,M_2+C_2M_4k_{02}\}.$$

若条件(H)成立，则可以去除定理 1 和定理 2 中的条件(W). 事实上，因为(H)成立，所以有

$$\sum_{t=1}^{T}[(h_1(t),\bar{u}_1+\tilde{u}_1(t))+(h_2(t),\bar{u}_2+\tilde{u}_2(t))]=\Big|\sum_{t=1}^{T}[(h_1(t),\tilde{u}_1(t))+(h_2(t),\tilde{u}_2(t))]\Big|\leqslant$$

$\|\tilde{u}_1\|_\infty \sum_{t=1}^{T} |h_1(t)| + \|\tilde{u}_2\|_\infty \sum_{t=1}^{T} |h_2(t)| \leqslant M_7 C(p') \cdot (\sum_{t=1}^{T} |\Delta u_1(t)|^p)^{1/p} + M_8 C(q')(\sum_{t=1}^{T} |\Delta u_2(t)|^q)^{1/q}$.

即(10)式中没有 $M_7|\bar{u}_1| + M_8|\bar{u}_2|$ 这一项. 进而在(6)式中也没有 $\max\{M_7, M_8\}(|\bar{u}_1| + |\bar{u}_2|)$ 这一项,以至于没有必要考虑 $W_1(|\bar{u}_1|) + W_2(|\bar{u}_2|)$ 与 $|\bar{u}_1| + |\bar{u}_2|$ 之间的关系,从而可以去除条件(W). 故可获得如下 2 个定理.

定理 3 假设$(\varphi_0)(a=+\infty)$,(Φ_0),(H),(F_1) ~ (F_3)(其中 $\alpha_1 \in [0, p-1)$ 且 $\alpha_2 \in [0, q-1)$)和(F_4)成立,则系统(2)在 E 中至少有 1 个 T-周期解.

定理 4 假设 $p=q$,$(\varphi_0)(a=+\infty)$,$(\Phi_0)'$,(H),(F_1),$(F_2)'$,(F_3)(其中 $\alpha_1 = p-1$ 且 $\alpha_2 = p-1$)和(F_4)条件成立,则系统(2)在 E 中至少有 1 个 T-周期解.

3 算例分析

仅验证注 4 中所给出的例子,注 2 中所给出的例子可以类似验证.

例 1 令 $\varphi_1(x) = 3A_1|x|^2 + 4|x|^3$,$\varphi_2(x) = 3A_2|x|^2 + 4|x|^3$,$p=3$,则易见$(\Phi_0)'$成立,显然 G 和 H 均满足条件(F_1),

$$G(t, x_1+x_2, y_1+y_2) = a_1(t)|x_1+x_2|^3 + a_2(t)|y_1+y_2|^3 \leqslant 4(a_1(t)|x_1|^3 + a_2(t)|y_1|^3) + 4(a_1(t)|x_2|^3 + a_2(t)|y_2|^3) = 4G(t, x_1, y_1) + 4G(t, x_2, y_2).$$

故 G 为(1,4)-次凸的,$\lambda = 1 > 1/2$,$\mu = 4 = 2^2 = 2^{p-1}\lambda^p$,从而 G 满足条件$(F_2)'$. 又

$$|\nabla_{x_1} H(t, x_1, x_2)| = 3b_1(t)|x_1|^2 + \frac{2|x_1|}{1+|x_1|^2} \cdot \ln(1+|x_2|^2) \leqslant 3b_1(t)|x_1|^2 + \ln(1+|x_2|^2).$$

同理有

$$|\nabla_{x_2} H(t, x_1, x_2)| \leqslant 3b_2(t)|x_2|^2 + \ln(1+|x_1|^2).$$

取 $\omega_1(t) = \omega_2(t) = 3t^2$,则 $W_1(t) = W_2(t) = t^3$,从而 $\widetilde{W}_1(t) = \max_{s\geqslant 0}\{ts - s^3\} = 2t^{3/2}/27 = \widetilde{W}_2(t)$. 故 $\widetilde{W}_1^{-1}(t) = \widetilde{W}_2^{-1}(t) = 9t^{2/3}/2^{2/3}$,从而

$$\widetilde{W}_1^{-1}(W_2(|x_2|)) = 9[W_2(|x_2|)]^{2/3}/2^{2/3} = 9[|x_2|^3]^{2/3}/2^{2/3} = 9|x_2|^2/2^{2/3}.$$

同理有 $\widetilde{W}_2^{-1}(W_1(|x_1|)) = 9|x_1|^2/2^{2/3}$.

又注意到 $\ln(1+|x_i|^2) < 9|x_i|^2/2^{2/3}$,从而

$$|\nabla_{x_1} H(t, x_1, x_2)| \leqslant 3b_1(t)|x_1|^2 + \frac{9}{2^{2/3}}|x_2|^2 = b_1(t)\omega_1(|x_1|) + \widetilde{W}_1^{-1}(W_2(|x_2|)), \quad (12)$$

$$|\nabla_{x_2} H(t, x_1, x_2)| \leqslant b_2(t)\omega_2(|x_2|) + \widetilde{W}_2^{-1}(W_1(|x_1|)). \quad (13)$$

取 $f_1(t) = b_1(t)$,$f_2(t) = b_2(t)$,$g_1(t) = g_2(t) = 1$,$l_1(t) = l_2(t) = 0$,$\forall t \in Z[1, T]$. 从而

$$M_1 = \sum_{t=1}^{T} f_1(t) = \sum_{t=1}^{T} b_1(t),\ M_2 = \sum_{t=1}^{T} g_1(t) = T,$$
$$M_3 = \sum_{t=1}^{T} l_1(t) = 0,\ M_4 = \sum_{t=1}^{T} f_2(t) = \sum_{t=1}^{T} b_2(t),$$
$$M_5 = \sum_{t=1}^{T} g_2(t) = T.$$

此外,由 $\omega_m(t) = 3t^2 (m=1,2)$ 易知,

$$\omega_m(s+t) = 3(s+t)^2 \leqslant 6(s^2+t^2) = 2(\omega_m(t) + \omega_m(t)),\ m=1,2.$$

故取 $C_m = 2$,$m=1,2$. 这样若取 $k_{11} = k_{21} = 3$,$k_{12} = k_{22} = 0$,$\alpha_1 = \alpha_2 = 2 = p-1$,则 $\omega_m(t)$ 符合(F_3)中的条件(i) ~ (v).

因此,(12)式和(13)式说明条件(F_3)成立. 又

$$L_1 = \frac{2M_1k_{11}C_1}{\alpha_1+1}C^{\alpha_1+1}(p') + \frac{M_2k_{11}}{\alpha_1+1}C^{\alpha_1+1}(p') + \frac{M_5C_1k_{11}}{\alpha_1+1}C^{\alpha_1+1}(p') = \left(\frac{12\sum_{t=1}^{T} b_1(t)}{3} + \frac{3T}{3} + \frac{6T}{3}\right) \cdot \left[C\left(\frac{3}{2}\right)\right]^3 = \left(4\sum_{t=1}^{T} b_1(t) + 3T\right)\left[C\left(\frac{3}{2}\right)\right]^3,$$

$$A_0 = \max_{t\in Z[1,T], |x_1|\leqslant 1, |x_2|\leqslant 1}(a_1(t)|x_1|^3 + a_2(t)|x_2|^3) \leqslant \max_{t\in Z[1,T]}(a_1(t) + a_2(t)),$$

$$L_3 = \frac{(2C_2M_4 + M_5)k_{21}}{\alpha_2+1}C^{\alpha_2+1}(q') = 3\left(4\sum_{t=1}^{T} b_2(t) + T\right)\left(C\left(\frac{2}{3}\right)\right)^3/p,$$

则易见$(\Phi_0)'$成立.

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The Existence of Periodic Solutions for a Class of Difference Systems with (φ_1,φ_2)-Laplacian

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Abstract: The existence of periodic solutions for a class of difference systems with a (φ_1,φ_2)-Laplacian is considered in this paper. By using the least action principle and definition of N-function and its properties,some existence criteria of periodic solutions are obtained under the condition that potential function has subconvex growth,(p,q)-sublinear growth and p-sublinear growth.

Key words: difference systems; (φ_1,φ_2)-Laplacian; the least action principle; periodic solutions

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