文章编号: 1000-5862(2012)04-0350-05

带退化粘性项的单个守恒律一般初边值问题的解的 L^p-收敛率

易菊燕

(暨南大学数学系，广东 广州 510632)

摘要: 在半空间中讨论具有一般边界的带退化粘性项的单个守恒律初边值问题的解的收敛率. 在流函数为凸条件下, 使用 L^1-估计导出了解渐近衰减到稀疏波的 1 个 L^p-衰减估计, 从而澄清了一般边界条件对衰减率的影响.

关键词: 退化粘性项; 一般初边值问题; 稀疏波; L^1-估计; 衰减估计

中图分类号: O 175.27 文献标志码: A

0 引 言

带退化粘性项的方程考虑如下单个守恒律的一般初边值问题

$$\begin{cases} u_t + f(u)_x = t^{\alpha} u_{xx}, \quad x>0, t>0, \\ u(x,0)=u_0(x)=\begin{cases} u_-, x=0, \\ u_+, x\to\infty, \end{cases} \\ u(0,t)=u_b(t), \quad t\geqslant 0, \end{cases} \tag{1}$$

其中 $u_\pm$ 为常数, f 为充分光滑的函数.

假设 f 为严格凸函数, 即存在正常数 α, 使得

$$f''(u)\geqslant \alpha>0, \tag{2}$$

且特征速度 $f'(u_\pm)$ 满足

$$0\leqslant f'(u_-)<f'(u_+), \tag{3}$$

其中边界条件假设为

$$\begin{aligned} &\lim_{t\to\infty} u_b(t)=u_-, u_b(0)=u_0(0), \\ &u_b(\cdot)-u_- \in H^1(\mathbf{R}_+)\bigcap L^1(\mathbf{R}_+). \end{aligned} \tag{4}$$

在全空间上考虑与(1)式相应的双曲守恒律 Riemann 问题:

$$\begin{cases} r_t + f(r)_x = 0, x\in \mathbf{R}, t>0, \\ r(x,-1)=r_0^R(x):=\begin{cases} u_-, x>0, \\ u_+, x>0, \end{cases} \end{cases}$$

其弱熵解是一个稀疏波

$$r(x,t)=\begin{cases} u_-, x\leqslant f'(u_-)(t+1), \\ (f')^{-1}\left(\dfrac{x}{t+1}\right), \\ f'(u_-)(t+1)\leqslant x\leqslant f'(u_+)(t+1), \\ u_+, f'(u_+)(t+1)\leqslant x. \end{cases} \tag{5}$$

当 $\alpha=0$ 时, 称(1)式为互相位置 Burgers 方程. 关于 Burgers 方程的初边值问题和 Cauchy 问题的解收敛到稀疏波的渐近性研究已有了不少的结果[1-3]. 当 $\alpha>0$ 时, 对带退化粘性项的单个守恒律所做的研究相对较少. 当 $\alpha\in(0,1/(4q)), q>3/2$ 时, 文献[4]证明了 Cauchy 问题存在整体光滑解, 并且相应解渐近收敛到弱的稀疏波. 当 $\alpha\in(0,1/7)$ 时, 文献[5]证明了 Cauchy 问题存在整体光滑解. 最近, 文献[6]证明了当 $\alpha\in(0,1/7)$ 时, 问题(1)整个解的存在性及解渐近衰减到 1 个稀疏波的 L^2-衰减估计, 而本文在此基础之上将衰减估计推广为 L^p-衰减估计, 并由此澄清了一般边界条件对解的衰减率的影响.

定理 1 如果(2)~(4)式成立, $u_0-u_+\in H^1\bigcap L^1(\mathbf{R}_+)$, $\alpha\in(0,1/7)$, 则问题(1)存在唯一整体解 $u(x,t)$ 满足估计式

$$\begin{aligned} &\|u(t)-r(t)\|_{L^p}\leqslant C\left(1+t\right)^{-(1-\alpha)/2+(\alpha+1)/2p}\cdot \\ &\sup_{0\leqslant\tau\leqslant t}\left(1+\left(1+\tau\right)b_0\right)^{1+1/p}+\left|u_- - u_b(t)\right|, \end{aligned} \tag{6}$$

收稿日期: 2012-01-06

基金项目: 国家自然科学基金(10871082)资助项目.

作者简介: 易菊燕(1987-), 女, 江西赣州人, 硕士研究生, 主要从事应用偏微分方程的研究.

其中 $2\leqslant p<\infty$ ， $b_0:=|u_- - u_b(\cdot)|+|u_b'(\cdot)|$ 及 C 是仅依赖于 u_0 和 p 的正常数.

特别地，若取 $|u_- - u_b(t)|$，$|u_b'(t)|=O(1)(1+t)^{-1}$，则有如下估计式

$$\|u(t)-r(t)\|_{L^p}\leqslant C(1+t)^{-(1-\alpha)/2+(\alpha+1)/2p}. \tag{7}$$

用 $C_{a,b}$ 表示仅依赖于 a,b 的一般正常数，或在不会混淆的情况下，简记 C ， $L^p=L^p((0,\infty))$ 表示一般 Lebesgue 空间，其范数为 $\|f\|_{L^p}$. $H^{m,p}=H^{m,p}((0,\infty))$ 表示一般的 Sobolev 空间，其范数为 $\|f\|_{H^{m,p}}$. 为简洁起见， 记 $H^m=H^{m,2}$, $C^k([0,T];\ H^p([0,1])$ $(T>0)$ 表示在 $[0,T]$ 上取值于 $H^p([0,1])$ 的 k 次连续可微函数空间，用 $\|\cdot\|$ 表示 $\|\cdot\|_{L^2}$，用 $\|\cdot\|_m$ 表示 $\|\cdot\|_{H^m}$.

1　光滑逼近及问题的转化

首先借助 Y. Hattori 和 K. Nishihara 的思想[7]来构造 $r(x,t)$ 的光滑逼近函数. 定义 $\tilde{\omega}(x,t)$ 为 Cauchy 问题

$$\begin{cases}\tilde{\omega}_t+\tilde{\omega}\tilde{\omega}_x=\tilde{\omega}_{xx}, & x\in\mathbf{R}, t>-1\\ \tilde{\omega}(x,-1)=\omega_0{}^R(x), x\in\mathbf{R}\end{cases} \tag{8}$$

的解，其中初始值 $\omega_0{}^R(x)$ 定义为

$$\omega_0{}^R(x)=\begin{cases}f'(u_-), x<0,\\ f'(u_+), x>0,\end{cases}(\ f'(u_-)>0\ ),$$

$$\omega_0{}^R(x)=\begin{cases}-f'(u_+), x<0,\\ f'(u_+), x>0,\end{cases}(\ f'(u_-)=0\ ).$$

由于(8)式为 Burgers 方程，所以利用 Hopf-Cole 变换可得 $\tilde{\omega}(x,t)$ 的显式表达式，从而能够定义稀疏波 $r(x,t)$ 的光滑近似 $\omega(x,t)$ 为

$$\omega(x,t)=(f')^{-1}(\tilde{\omega}(x,t)).$$

接下来，对边界进行修正，定义修正光滑逼近函数 $W(x,t)$ 为

$$W(x,t)=\omega(x,t)-\psi(x,t),$$

其中 $\psi(x,t):=(\omega(0,t)-u_b(t))\mathrm{e}^{-x}$.

下面定义扰动 $v(x,t):=u(x,t)-W(x,t)$ ，则问题(1)转化为

$$\begin{cases}v_t+(f(W+v)-f(W))_x=t^\alpha v_{xx}+\left(t^\alpha-1\right)W_{xx}+\\ Q(x,t), x>0, t>0,\\ v(x,0)=v_0(x):=u_0(x)-W_0(x)\to 0\ (x\to\infty),\\ x>0, v(0,t)=0, t\geqslant 0,\end{cases} \tag{9}$$

其中

$$Q(x,t)=-\frac{f'''(\omega)}{f''(\omega)}\omega_x^2+\psi_t+(f(W+\psi)-f(W))_x-\psi_{xx}.$$

引理 1　对于 $1\leqslant p\leqslant\infty$ ， $t\geqslant 0$ ， $W(x,t)$ 和 $Q(x,t)$ 满足

(i) $\|W(t)-r(t)\|_p\leqslant C(1+t)^{-1/2+1/(2p)}+|u_b(t)-u_-|$;

(ii) $\|W_x(t)\|_p\leqslant C(1+t)^{-1+1/p}+|u_b(t)-u_-|$,

$\|W_{xx}(t)\|_p\leqslant C(1+t)^{-3/2+1/(2p)}+|u_b(t)-u_-|$;

(iii) $W_x(x,t)=\phi(x,t)+(u_- - u_b(t))\mathrm{e}^{-x}$,

其中 $\phi(x,t):=\omega_x(x,t)+(\omega(0,t)-u_-)\mathrm{e}^{-x}>0,\ x\in\mathbf{R}_+$;

(iv) $\|Q(t)\|_p\leqslant C(1+t)^{-2+1/p}+|u_b'(t)|+|u_b(t)-u_-|$.

由于 $W(x,t)$ 非常快地收敛到(5)式的稀疏波 $r(x,t)$，因此要证估计式(6)和(7)，只需导出 $v(x,t)$ 的衰减估计.

2　衰减估计

对于 $0<T\leqslant\infty$，定义解空间

$$X(0,T)=\Big\{v(x,t)\Big|v\in C^0\left([0,T];H_0^1(\mathbf{R}_+)\right),$$
$$t^{\alpha/2}v_x\in L^2\left(0,T;H^1(\mathbf{R}_+)\right)\Big\},$$

为了得到问题(9)解的 L^p -衰减估计，需要建立 $v(x,t)$ 的如下 L^1-估计.

引理 2　假设 $v_0\in\left(H^1\cap L^1\right)\left(\mathbf{R}^+\right)$ 且 $\alpha\in(0,1/7)$，则问题(9)的解 $v(x,t)$ 满足如下估计式

$$\|v(t)\|_{L^1}\leqslant\|v_0\|_{L^1}+C(1+t)^\alpha\sup_{0\leqslant\tau\leqslant t}\left(1+(1+\tau)b_0\right).$$

下面取得 v 的衰减估计. 由引理 1 结合如下定理即可证得定理 1. 下面定理的证明思路与文献[8-9]类似.

定理 2(衰减估计)　假设条件(2)~(4)成立，若 $v_0\in\left(H^1\cap L^1\right)\left(\mathbf{R}^+\right)$ 且 $\alpha\in(0,1/7)$ ，则问题(9)的解 $v(x,t)$, $\forall\varepsilon\in(0,1/7)$，满足下列估计式

$$(1+t)^{1/2-3\alpha/2+\varepsilon}\|v\|_{L^p}^p+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\left(\tau^\alpha\|\tilde{v}_x\|^2+\right.$$
$$\left.\|\sqrt[p]{\varphi}v\|_{L^p}^p\right)\mathrm{d}\tau\leqslant$$
$$C(1+t)^{\varepsilon+1-\alpha+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}(1+(1+\tau)b_0)^{p+1},$$

其中 $\tilde{v}:=|v|^{p/2-1}v$ ， $b_0:=|u_- - u_b(\cdot)|+|u_b'(\cdot)|$ ， 且 $2\leqslant p<\infty$. 特别地，若取 $|u_- - u_b(t)|$，$|u_b'(t)|=O(1)(1+t)^{-1}$，

则有如下估计式

$$\|v\|_{L^p}\leqslant C(1+t)^{-(1-\alpha)/2+(\alpha+1)/(2p)}.$$

证 在方程(9)两边同乘以 $|v|^{p-2}v$ 得

$$\frac{1}{p}\left(|v|^p\right)_t+\left(F-t^\alpha|v|^{p-2}vv_x\right)_x+\frac{4(p-1)}{p^2}t^\alpha\left|\left(|v|^{p/2-1}v\right)_x\right|^2+(p-1)\int_0^v(f'(W+\eta)-f'(W))|\eta|^{p-2}\mathrm{d}\eta W_x=Q|v|^{p-2}v+(t^\alpha-1)W_{xx}|v|^{p-2}v,\tag{10}$$

其中 $F:=\left(f(W+v)-f(W)\right)|v|^{p-2}v-(p-1)\int_0^v\left(f(W+\eta)-f(W)\right)|\eta|^{p-2}\mathrm{d}\eta$.

由于 $v(0,t)=0$ 可得, (10)式第 2 项关于 x 在 $(0,\infty)$ 上积分为 0, 故将(10)式关于 x 在 $(0,\infty)$ 上积分得

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_p^p+\frac{4(p-1)t^\alpha}{p^2}\left\|\left(|v|^{p/2-1}v\right)_x\right\|^2+(p-1)\int_0^\infty\int_0^v\left(f'(W+\eta)-f'(W)\right)|\eta|^{p-2}\mathrm{d}\eta W_x\,\mathrm{d}x=\int_0^\infty Q|v|^{p-2}v\,\mathrm{d}x+\int_0^\infty(t^\alpha-1)W_{xx}|v|^{p-2}v\,\mathrm{d}x.\tag{11}$$

由 W 及 W_x 的有界性, (11)式左边第 3 项利用(2)式及引理 1(iii)可得

$$(p-1)\int_0^\infty\int_0^v\left(f'(W+\eta)-f'(W)\right)|\eta|^{p-2}\mathrm{d}\eta W_x\,\mathrm{d}x\geqslant\frac{\beta(p-1)}{2p}\int_0^\infty\varphi|v|^p\mathrm{d}x-C\int_0^\infty|u_--u_b(t)|\mathrm{e}^{-x}|v|^p\mathrm{d}x.\tag{12}$$

将估计式(12)代入(11)式, 并在 $(0,\infty)$ 上关于 x 积分得

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_p^p+\frac{\beta(p-1)}{2p}\int_0^\infty\varphi|v|^p\mathrm{d}x+\frac{4(p-1)t^\alpha}{p^2}\cdot\left\|\left(|v|^{p/2-1}v\right)_x\right\|^2\leqslant C\int_0^\infty|u_--u_b(t)|\mathrm{e}^{-x}|v|^p\mathrm{d}x+\int_0^\infty Q|v|^{p-2}v\,\mathrm{d}x+\int_0^\infty(t^\alpha-1)W_{xx}|v|^{p-2}v\,\mathrm{d}x\leqslant C|u_--u_b(t)|\|v\|_{L^p}^p+\|Q\|_{L^1}\|v\|_\infty^{p-1}+(t^\alpha-1)\|W_{xx}\|_{L^1}\|v\|_\infty^{p-1}.\tag{13}$$

将(13)式乘以 $(1+t)^{1/2-3\alpha/2+\varepsilon}$ 并在 $(0,t)$ 上积分, 令 $\tilde{v}=|v|^{p/2-1}v$ 得

$$(1+t)^{1/2-3\alpha/2+\varepsilon}\|v\|_{L^p}^p+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}t^\alpha\|\tilde{v}_x\|^2\mathrm{d}\tau+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\left\|\sqrt[p]{\varphi}v\right\|_{L^p}^p\mathrm{d}\tau\leqslant C\left(\|v_0\|_{L^p}^p+\left(\frac{1-3\alpha}{2}+\varepsilon\right)\cdot\int_0^t(1+\tau)^{\varepsilon-(1-3\alpha)/2}\|v\|_{L^p}^p\,\mathrm{d}\tau+\int_0^t(1+\tau)^{(1-3\alpha)/2+\varepsilon}\|v\|_\infty^{p-1}\|Q\|_{L^1}\,\mathrm{d}\tau+\int_0^t(1+\tau)^{\frac{1}{2}-\frac{3}{2}\alpha+\varepsilon}\left(\tau^\alpha-1\right)\|v\|_\infty^{p-1}\|W_{xx}\|_{L^1}\,\mathrm{d}\tau+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}|u_--u_b(\tau)|\|v\|_{L^p}^p\,\mathrm{d}\tau\right):=C\left(\|v_0\|_{L^p}^p+I_1+I_2+I_3+I_4\right).\tag{14}$$

利用插值不等式 $\|v\|_\infty\leqslant C\|\tilde{v}_x\|^{2/(p+1)}\|v\|_{L^1}^{1/(p+1)}$, $\|v\|_{L^p}^p\leqslant C\|\tilde{v}_x\|^{2(p-1)/(p+1)}\|v\|_{L^1}^{2p/(p+1)}$ ($2<p<\infty$)及 Young 不等式, 估计 I_1 得

$$I_1\leqslant C\left(\frac{1}{2}-\frac{3}{2}\alpha+\varepsilon\right)\int_0^t(1+\tau)^{\varepsilon-1/2-3\alpha/2}\|\tilde{v}_x\|^{2(p-1)/(p+1)}\cdot\|v\|_{L^1}^{2p/(p+1)}\mathrm{d}\tau\leqslant\frac{1}{8}\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|\tilde{v}_x\|^2\mathrm{d}\tau+C\int_0^t(1+\tau)^{\varepsilon-3\alpha/2-p/2}\tau^{(1-p)\alpha/2}\|v\|_{L^1}^p\mathrm{d}\tau.\tag{15}$$

同理, 估计 I_2,I_3,I_4 得

$$I_2\leqslant C\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\|Q\|_{L^1}\|\tilde{v}_x\|^{2(p-1)/(p+1)}\cdot\|v\|_{L^1}^{(p-1)/(p+1)}\mathrm{d}\tau\leqslant\frac{1}{8}\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|\tilde{v}_x\|^2\mathrm{d}\tau+C\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{(1-p)\alpha/2}\|Q\|_{L^1}^{(p+1)/2}\|v\|_{L^1}^{(p-1)/2}\mathrm{d}\tau,\tag{16}$$

$$I_3\leqslant\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|v\|_\infty^{p-1}\|W_{xx}\|_{L^1}\mathrm{d}\tau\leqslant C\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|\tilde{v}_x\|^{2(p-1)/(p+1)}\|v\|_{L^1}^{(p-1)/(p+1)}\cdot\|W_{xx}\|_{L^1}\mathrm{d}\tau\leqslant\frac{1}{8}\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|\tilde{v}_x\|^2\mathrm{d}\tau+C\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|v\|_{L^1}^{(p-1)/2}\|W_{xx}\|_{L^1}^{(p+1)/2}\mathrm{d}\tau,\tag{17}$$

$$I_4\leqslant C\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}|u_--u_b(\tau)|\|\tilde{v}_x\|^{2(p-1)/(p+1)}\cdot\|v\|_{L^1}^{2p/(p+1)}\mathrm{d}\tau\leqslant\sup_{0\leqslant\tau\leqslant t}|u_--u_b(\cdot)|\left[\varepsilon_1\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\cdot\|\tilde{v}_x\|^2\,\mathrm{d}\tau+C\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{(1-p)\alpha/2}\|v\|_{L^1}^p\,\mathrm{d}\tau\right]\leqslant\varepsilon_1\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\|\tilde{v}_x\|^2\,\mathrm{d}\tau+C\sup_{0\leqslant\tau\leqslant t}|u_--u_b(\cdot)|\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{(1-p)\alpha/2}\|v\|_{L^1}^p\,\mathrm{d}\tau.\tag{18}$$

取 ε_1 足够小, 将估计式(15)~(18)代入(14)式可得

$$(1+t)^{1/2-3\alpha/2+\varepsilon}\|v\|_{L^p}^p+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}t^\alpha\|\tilde{v}_x\|^2\mathrm{d}\tau+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\left\|\sqrt[p]{\varphi}v\right\|_{L^p}^p\mathrm{d}\tau\leqslant C\left(\|v_0\|_{L^p}^p+\int_0^t(1+\tau)^{\varepsilon-3\alpha/2-p/2}\tau^{(1-p)\alpha/2}\|v\|_{L^1}^p\mathrm{d}\tau+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{(1-p)\alpha/2}\|Q\|_{L^1}^{(p+1)/2}\|v\|_{L^1}^{(p-1)/2}\mathrm{d}\tau+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^\alpha\cdot\|v\|_{L^1}^{(p-1)/2}\|W_{xx}\|_{L^1}^{(p+1)/2}\mathrm{d}\tau+\sup_{0\leqslant\tau\leqslant t}|u_--u_b(\cdot)|\cdot\right.$$

$$\int_0^t (1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{(1-p)\alpha/2}\|v\|_{L^1}^p\,\mathrm{d}\tau\Big):=$$

$$C\left(\|v_0\|_{L^p}^p + J_1+J_2+J_3+J_4\right). \tag{19}$$

由引理2估计 J_1 并定义 $b_0:=|u_- - u_b(\cdot)|+|u_b'(\cdot)|$，得

$$J_1\leqslant\int_0^t(1+\tau)^{\varepsilon-\alpha-(\alpha+1)p/2}\Big(\|v_0\|_{L^1}+C(1+\tau)^{\alpha}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)\Big)^p\mathrm{d}\tau\leqslant$$
$$C\int_0^t(1+\tau)^{\varepsilon-\alpha-(\alpha+1)p/2}(1+\tau)^{\alpha p}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p\mathrm{d}\tau\leqslant$$
$$C\int_0^t(1+\tau)^{\varepsilon-\alpha+(\alpha+1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p\mathrm{d}\tau\leqslant$$
$$C(1+t)^{\varepsilon+1-\alpha+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p. \tag{20}$$

同理估计 J_3， J_4 得

$$J_3\leqslant\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{\alpha}\Big((1+\tau)^{-1}+|u_b(\cdot)-u_-|\Big)^{(p+1)/2}\cdot\|v\|_{L^1}^{(p-1)/2}\mathrm{d}\tau\leqslant\int_0^t(1+\tau)^{\varepsilon-\alpha+(\alpha-1)p/2}\big(1+(1+\tau)\cdot$$
$$|u_b(\cdot)-u_-|\big)^{(p+1)/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^{(p-1)/2}\mathrm{d}\tau\leqslant$$
$$\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p\cdot\int_0^t(1+\tau)^{\varepsilon-\alpha+(\alpha-1)p/2}\mathrm{d}\tau\leqslant$$
$$C(1+t)^{\varepsilon+1-\alpha+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p, \tag{21}$$

$$J_4\leqslant\sup_{0\leqslant\tau\leqslant t}|u_- - u_b(\tau)|\int_0^t(1+\tau)^{\varepsilon+1/2-(1+p/2)\alpha}\Big(\|v_0\|_{L^1}+C(1+\tau)^{\alpha}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)\Big)^p\mathrm{d}\tau\leqslant C\sup_{0\leqslant\tau\leqslant t}|u_- - u_b(\tau)|\cdot$$
$$\int_0^t(1+\tau)^{\varepsilon+1/2-(1+p/2)\alpha}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p\mathrm{d}\tau\leqslant$$
$$C(1+t)^{\varepsilon+3/2-(1+p/2)\alpha}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^{p+1}. \tag{22}$$

最后由引理 1(iv)及引理 2 估计 J_2，得

$$J_2\leqslant\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\tau^{(1-p)\alpha/2}\Big((1+\tau)^{-1}+|u_b(\cdot)-u_-|+|u_b'(\cdot)|\Big)^{(p+1)/2}\|v\|_{L^1}^{(p-1)/2}\mathrm{d}\tau\leqslant\int_0^t(1+\tau)^{\varepsilon-3\alpha/2-p/2}\cdot$$
$$\big(1+(1+\tau)b_0\big)^{(p+1)/2}\cdot\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^{(p-1)/2}\mathrm{d}\tau\leqslant$$
$$\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p\cdot\int_0^t(1+\tau)^{\varepsilon-3\alpha/2-p/2}\mathrm{d}\tau\leqslant$$
$$C(1+t)^{\varepsilon+1-3\alpha/2-p/2}\sup_{0\leqslant\tau\leqslant t}(1+(1+\tau)b_0)^p. \tag{23}$$

将估计式(20)~(23)代入(19)式得

$$(1+t)^{1/2-3\alpha/2+\varepsilon}\|v\|_{L^p}^p+\int_0^t(1+\tau)^{1/2-3\alpha/2+\varepsilon}\Big(\tau^{\alpha}\|\tilde v_x\|^2+\|\sqrt[p]{\varphi}v\|_{L^p}^p\Big)\mathrm{d}\tau\leqslant C(1+t)^{\varepsilon+1-\alpha+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p+$$
$$C(1+t)^{\varepsilon+1-3\alpha/2-p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p+$$
$$C(1+t)^{\varepsilon+1-\alpha+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^p+$$
$$C(1+t)^{\varepsilon+3/2-\alpha-p\alpha/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^{p+1}\leqslant$$
$$C(1+t)^{\varepsilon+1-\alpha+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^{p+1},$$

特别地，

$$\|v\|_{L^p}^p\leqslant C(1+t)^{(\alpha+1)/2+(\alpha-1)p/2}\sup_{0\leqslant\tau\leqslant t}\big(1+(1+\tau)b_0\big)^{p+1}.$$

若取 $|u_- - u_b(t)|$， $|u_b'(t)|=O(1)(1+t)^{-1}$，由上式得

$$\|v\|_{L^p}\leqslant C(1+t)^{-(1-\alpha)/2+(\alpha+1)/2p}.$$

至此，定理 2 得证.

3　参考文献

[1] Harabetian E. Rarefactions and large time behavior for parabolic equations and monotone schemes [J]. Commum Math Phys, 1988, 114(4): 527-536.

[2] II'in A M, Oleinik O A. Behavior of the solution of the Cauchy problem for certain quasilinear equations for unbounded increase of the time [J]. AMS Transl, 1964, 42(2): 19-23.

[3] Kawashima S, Nishibata S, Nishikawa M. Asymptotic stability of stationary waves for two-dimensional viscous conservation laws in half space [J]. Discrete and Continuous Dynamical Systerms, 2003(1): 469-476.

[4] Xu Yanling, Jiang Mina. Asymptiotic stability of rarefaction wave for generalized Burgers equation [J]. Acta Math Sci, 2005, 25B(1): 119-129.

[5] Chen Jing, Zhu Changjiang. Decay rates of strong planar rarefaction waves to scalar conservation laws with degenerate viscosity in several dimensions [J]. Trans Amer Math Soc, 2010, 362(4): 1797-1830.

[6] Feng Ruirui, Li Hailing, Zhong Weiming. L^2-decay rate to rarefaction waves for general initial-boundary value probelm of scalar viscous conservation laws [J]. Far East Journal of Math Sci, 2010, 42(2): 205-218.

[7] Hattori Y, Nishihara K. A note on the stability of the rarefaction wave of the Burgers equation [J]. Japan J Indust Appl Math, 1991, 8(1): 85-96.

[8] Ito K. Asymptotic decay toward the planar rarefaction waves of solutions for viscous conservation laws in several space dimensions [J]. Math Models Methods Appl Sci, 1996, 6(3): 315-338

[9] Nakamura T. Asymptotic decay toward the rarefaction waves of solutions for viscous conservations laws in a one–dimensional half space [J]. SIAM J Math Anal, 2003, 34(6): 1308-1317.

The L^p-Convergence Rate for Initial-Boundary Value Problem of Scalar Conservation Law with Degenerate Viscosity

YI Ju-yan

(Department of Mathematics, Ji’nan University, Guangzhou Guangdong 510632, China)

Abstract: It is concerned with convergence rates toward the rarefaction waves of the solutions of the initial boundary data problem for scalar conservation law with degenerate viscosity and the general boundary data. Under the condition of convex flux, using L^1-estimate derives a L^p-decay rate of the rarefaction wave for scalar conservation law with degenerate viscosity. From this decay rate estimate, the effect of the general boundary data on the decay rate is clarified.

Key words: degenerate viscosity; general initial-boundary value problem; rarefaction wave; L^1-estimate; decay rate

(责任编辑：曾剑锋)

(上接第 338 页)

On the Growth for Solutions of a Certain Higher Order Differential Equation

FENG Bin, LIU Hui-fang*, LI Yan-ling

(College of Mathematics and Informatics, Jiangxi Normal University, Nanchang Jiangxi 330022, China)

Abstract: The growth for solutions of a differential equation $f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_2 f'' + A_1 e^{az^n} f' + A_0 e^{bz^n} f = F$ has been investigated, where $A_0(z)$, $A_1(z)$ and $F(z)$ are entire functions with order less than *n*, $A_j(z)$ $(j = 2,3,\cdots,k-1)$ are polynomials with degree no more than *m*, *a* and *b* are nonzero complex numbers, then every solution $f(z)$ of the above equation satisfies

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \infty, \overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = n,$$

except at most two exceptional complex numbers *b*.

Key words: differential equation; order of growth; exponent of convergence; hyper-order

(责任编辑：王金莲)