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Fractional Pfaff-Birkhoff Principle and Birkhoff's Equations within Caputo Fractional Derivatives

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Abstract: The fractional Pfaff-Birkhoff variational problem is studied under Caputo fractional derivative. First, the definition of Caputo fractional derivatives, the formula for integration by parts and the commutative relations between differential operation and variational operation are given. Second, the fractional Pfaff-Birkhoff principle and the fractional Birkhoff's equations are obtained. And finally an example is given to illustrate the application of the results.

Key words: fractional Pfaff-Birkhoff principle; fractional Birkhoff's equation; Caputo fractional derivative; transversality condition

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0 Introduction

Fractional Calculus first appeared in the letter that L'Hospital wrote to Leibniz asking about the nth-derivative of a function in 1 695. If $n = 1/2$, what would the result will be, then the fractional calculus was born. The study about fractional calculus is first in mathematics, Euler, Laplace and Fourier do some works on it. But the development of the fractional calculus is slowly, until 1974 the first book about fractional calculus was published [1]. In recent decades, the application of the factional calculus was used in many fields, such as physics, chemistry, biology, electronics, economics, control systems and so on [2-3].

There are various types about fractional derivative's definitions, including Riemann-Liouville, Caputo, Riesz, Hilfer, Jumarie, Hadamard, Grünwald-Letnikov and others. But in application, the Riemann-Liouville and Caputo definitions were mainly used. Compared with Riemann-Liouville definition, the Caputo derivative was more widely used. Because the initial conditions for fractional differentional equations with Caputo derivatives take on the same form as for integer-order differential equations, but it is not the same for Riemann-Liouville derivatives and it should use fractional initial conditions. On the other hand, the Caputo derivative of a constant is zero, while the Riemann-Liouville derivative is not.

In recent decades, the development of the fractional variational problem was going well. F. Riewe [4-5] first apply the fractional calculus in the nonconservative mechanical system. After that, the fractional variational problem was studied by many scholars. O. P. Agrawal [6-8] considered the simplest fractional variational problems and Lagrange fractional variational problems within different fractional derivatives, he derived the corresponding fractional Euler-Lagrange heorems and discussed the boundary conditions' possibility of every situation. Different from O. P. Agrawal's, T. M. Atanacković [9-11] considered a fractional variational problem which the integration's lower bound of the functional is different from the fractional derivative's lower bound of the Lagrangian. A. R. El-Nabulsi [12-15] introduced a factor, then he considered a new fractional variational problems which called fractional action-like

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variational problems. Moreover, S. I. Muslih and D. Baleanu studied the fractional Euler-Lagrange equations and fractional Hamilton equations [16-18]. A. B. Malinowska and D. F. M. Torres studied the generalized natural boundary conditions of the fractional variational problems within Caputo derivative [19]. G. S. F. Frederico and D. F. M. Torres were the first to study the fractional invariable and conserved quantity, they established the fractional Noether theorem within Riemann-Liouville [20], Caputo [21], Riesz-Caputo [22] derivatives. They also studied the Noether theorem of the nonconservative system under the fractional action-like frame [23], and generalized the situation to the Lagrange function which contains higher order derivative [24]. E. M. Rabei and his cooperators studied the fractional Hamilton-Jacobi equations [25-27].

These scholars did a series of work about fractional variational problems, but the works just come down to the Lagrangian system and Hamiltonian system, not including Birkhoffian system. A Birkhoffian mechanics is a natural generalization of a Hamiltonian mechanics. It has been gained rich fruit through one century's study, The dynamics of Birkhoffian system becomes one of the hottest research directions in applied mathematics, physics and dynamics, but these study only come down to integer situation, not come down to fractional situation so far.

The fractional Birkhoffian system is studied in this paper, the fractional Pfaff-Birkhoff variational problem is studied within Caputo derivatives. The fractional Pfaff-Birkhoff principle and fractional Birkhoff's equations with the transversality condition are gained.

1 Preliminaries

In this section we briefly review some basic definitions and properties of fractional derivatives used in the following sections, and present the commutative relations between differential operation and variational operation and the formulae for fractional integration by parts in terms of Caputo derivatives. For more on the subject we refer the reader to literature [2-3].

Let f be a continuous and integrable function in the interval $[t_1, t_2]$. Then the left Riemann-Liouville fractional derivative is defined as

$${}_{t_1}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t_1}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau,$$

and the right Riemann-Liouville fractional derivative is defined as

$${}_{t}D_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t}^{t_2} \frac{f(\tau)}{(\tau-t)^{\alpha-m+1}} \mathrm{d}\tau,$$

where $\Gamma(*)$ is Gamma function, α is the order of the fractional derivative and it fulfills $m-1 \leqslant \alpha < m$.

The left Caputo fractional derivative is defined as

$${}_{t_1}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_1}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau,$$

$$m-1 \leqslant \alpha < m,$$

and the right Caputo fractional derivative is defined as

$${}_{t}^{C}D_{t_2}^{\alpha} f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{t}^{t_2} \frac{f^{(m)}(\tau)}{(\tau-t)^{\alpha-m+1}} \mathrm{d}\tau,$$

$$m-1 \leqslant \alpha < m.$$

If α is an integer, then we have

$${}_{t_1}D_t^{\alpha} f(t) = {}_{t_1}^{C}D_t^{\alpha} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t),$$

$${}_{t}D_{t_2}^{\alpha} f(t) = {}_{t}^{C}D_{t_2}^{\alpha} f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} f(t).$$

Let f and g be two smooth functions on $[t_1, t_2]$. Then the formula for fractional integration by parts in terms of Caputo derivative [7] is

$$\int_{t_1}^{t_2} g(t)\left({}_{t_1}^{C}D_t^{\alpha} f(t)\right) \mathrm{d}t = \int_{t_1}^{t_2} f(t)\left({}_{t}D_{t_2}^{\alpha} g(t)\right) \mathrm{d}t + \sum_{k=0}^{m-1} {}_{t}D_{t_2}^{\alpha+k-m} g(t) \left.\frac{\mathrm{d}^{m-1-k} f(t)}{\mathrm{d}t^{m-1-k}}\right|_{t_1}^{t_2}, \tag{1}$$

$$\int_{t_1}^{t_2} g(t)\left({}_{t}^{C}D_{t_2}^{\alpha} f(t)\right) \mathrm{d}t = \int_{t_1}^{t_2} f(t)\left({}_{t_1}D_{t}^{\alpha} g(t)\right) \mathrm{d}t + \sum_{k=0}^{m-1} (-1)^{m+k} {}_{t_1}D_{t}^{\alpha+k-m} g(t) \left.\frac{\mathrm{d}^{m-1-k} f(t)}{\mathrm{d}t^{m-1-k}}\right|_{t_1}^{t_2}. \tag{2}$$

The commutative relations between differential operation and variational operation within Caputo derivatives are

$$\delta\, {}_{t_1}^{C}D_t^{\alpha} f = {}_{t_1}^{C}D_t^{\alpha} \delta f, \tag{3}$$

$$\delta\, {}_{t}^{C}D_{t_2}^{\alpha} f = {}_{t}^{C}D_{t_2}^{\alpha} \delta f. \tag{4}$$

Here, we give a proof [28].

Let

$$\delta f = f(t, \gamma + \mathrm{d}\gamma) - f(t, \gamma). \tag{5}$$

Expanding $f(t, \gamma + \mathrm{d}\gamma)$ and accurate to linear part of $\mathrm{d}\gamma$, we obtain

$$f(t, \gamma + \mathrm{d}\gamma) = f(t, \gamma) + \frac{\partial f(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma. \tag{6}$$

Substituting (6) into (5), we have

$$\delta f = \frac{\partial f(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma.$$

Then, we have

$${}_{t_1}^{C}D_t^{\alpha} \delta f = {}_{t_1}^{C}D_t^{\alpha}\left(\frac{\partial f(t, \gamma)}{\partial \gamma} \mathrm{d}\gamma\right), \tag{7}$$

$$\delta_{t_1}^{C} D_t^{\alpha} f = {}_{t_1}^{C} D_t^{\alpha} f(t,\gamma + \mathrm{d}\gamma) - {}_{t_1}^{C} D_t^{\alpha} f(t,\gamma) = {}_{t_1}^{C} D_t^{\alpha} [f(t,\gamma + \mathrm{d}\gamma) - f(t,\gamma)].$$

Using (6), we get

$$\delta_{t_1}^{C} D_t^{\alpha} f = {}_{t_1}^{C} D_t^{\alpha} \left(\frac{\partial f(t,\gamma)}{\partial \gamma} \mathrm{d}\gamma \right). \tag{8}$$

Hence, combining (7) and (8), we obtain the commutative relation (3).

We can get the commutative relation (4) in the same way.

2 Fractional Pfaff-Birkhoff principle and Birkhoff's equations within Caputo derivatives

The integral

$$A = \int_{t_1}^{t_2} \left\{ \sum_{\nu=1}^{2n} R_\nu^\alpha(t,\boldsymbol{a})\, {}_{t_1}^{C} D_t^\alpha a^\nu + \sum_{\nu=1}^{2n} R_\nu^\beta(t,\boldsymbol{a})\, {}_{t}^{C} D_{t_2}^\beta a^\nu - B(t,\boldsymbol{a}) \right\} \mathrm{d}t \tag{9}$$

is called the fractional Pfaff-Birkhoff action. Where $B(t,\boldsymbol{a})$ is Birkhoffian, $R_\nu^\alpha(t,\boldsymbol{a})$ and $R_\nu^\beta(t,\boldsymbol{a})$ $(\nu = 1,\cdots,2n)$ are Birkhoff's functions, $\boldsymbol{a}$ is Birkhoff variable. We also have $p-1 \leqslant \alpha < p, q-1 \leqslant \beta < q$ (p and q are integer).

The isochronous variational principle

$$\delta A = 0 \tag{10}$$

with the commutative relations

${}_{t_1}^{C} D_t^\alpha \delta a^\nu = \delta_{t_1}^{C} D_t^\alpha a^\nu, {}_{t}^{C} D_{t_2}^\beta \delta a^\nu = \delta_t^C D_{t_2}^\beta a^\nu (\nu = 1,\cdots,2n)$

and fixed endpoint conditions

$$\delta a^\nu |_{t=t_1} = \delta a^\nu |_{t=t_2} = 0 (\nu = 1,\cdots,2n) \tag{11}$$

can be called the fractional Pfaff-Birkhoff principle within Caputo derivatives.

The fractional Birkhoff's equations can be deduced from the fractional Pfaff-Birkhoff principle [29-30]. Expanding the principle (10), we have

$$\delta A = \int_{t_1}^{t_2} \left\{ \sum_{\nu=1}^{2n} \left(\delta R_{\nu\, t_1}^{\alpha\, C} D_t^\alpha a^\nu + R_\nu^\alpha \delta_{t_1}^C D_t^\alpha a^\nu + \delta R_{\nu\, t}^{\beta\, C} D_{t_2}^\beta a^\nu + R_\nu^\beta \delta_t^C D_{t_2}^\beta a^\nu \right) - \delta B \right\} \mathrm{d}t = 0,$$

i. e

$$\delta A = \int_{t_1}^{t_2} \left\{ \sum_{\mu=1}^{2n} \left[\sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu^\alpha}{\partial a^\mu}\, {}_{t_1}^{C} D_t^\alpha a^\nu + \frac{\partial R_\nu^\beta}{\partial a^\mu}\, {}_{t}^{C} D_{t_2}^\beta a^\nu \right) - \frac{\partial B}{\partial a^\mu} \right] \delta a^\mu + \sum_{\nu=1}^{2n} \left(R_\nu^\alpha \delta_{t_1}^C D_t^\alpha a^\nu + R_\nu^\beta \delta_t^C D_{t_2}^\beta a^\nu \right) \right\} \mathrm{d}t = 0. \tag{12}$$

Using the formulae (1) and (2) for integration by parts, we have

$$\int_{t_1}^{t_2} R_\nu^\alpha \delta({}_{t_1}^{C} D_t^\alpha a^\nu)\, \mathrm{d}t = \int_{t_1}^{t_2} R_{\nu\, t_1}^{\alpha\, C} D_t^\alpha \delta a^\nu \mathrm{d}t = \int_{t_1}^{t_2} \delta a^\nu\, {}_t D_{t_2}^\alpha R_\nu^\alpha \mathrm{d}t + \sum_{k=0}^{m-1} {}_t D_{t_2}^{\alpha+k-m} R_\nu^\alpha \frac{\mathrm{d}^{m-1-k}(\delta a^\nu)}{\mathrm{d}t^{m-1-k}} \bigg|_{t_1}^{t_2}, \tag{13}$$

$$\int_{t_1}^{t_2} R_\nu^\beta \delta({}_{t}^{C} D_{t_2}^\beta a^\nu)\, \mathrm{d}t = \int_{t_1}^{t_2} R_{\nu\, t}^{\beta\, C} D_{t_2}^\beta \delta a^\nu \mathrm{d}t = \int_{t_1}^{t_2} \delta a^\nu\, {}_{t_1} D_t^\beta R_\nu^\beta \mathrm{d}t + \sum_{k=0}^{m-1} (-1)^{m+k} \cdot {}_{t_1} D_t^{\beta+k-m} R_\nu^\beta \frac{\mathrm{d}^{m-1-k}(\delta a^\nu)}{\mathrm{d}t^{m-1-k}} \bigg|_{t_1}^{t_2}. \tag{14}$$

Substituting (13) and (14) into (12), we have

$$\delta A = \int_{t_1}^{t_2} \left\{ \sum_{\mu=1}^{2n} \left\{ \sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu^\alpha}{\partial a^\mu}\, {}_{t_1}^{C} D_t^\alpha a^\nu + \frac{\partial R_\nu^\beta}{\partial a^\mu}\, {}_{t}^{C} D_{t_2}^\beta a^\nu \right) - \frac{\partial B}{\partial a^\mu} + {}_t D_{t_2}^\alpha R_\mu^\alpha + {}_{t_1} D_t^\beta R_\mu^\beta \right\} \delta a^\mu \right\} \mathrm{d}t + \sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} {}_t D_{t_2}^{\alpha+k-m} R_\nu^\alpha \frac{\mathrm{d}^{m-1-k}(\delta a^\nu)}{\mathrm{d}t^{m-1-k}} \bigg|_{t_1}^{t_2} + \sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} (-1)^{m+k} \cdot {}_{t_1} D_t^{\beta+k-m} R_\nu^\beta \frac{\mathrm{d}^{m-1-k}(\delta a^\nu)}{\mathrm{d}t^{m-1-k}} \bigg|_{t_1}^{t_2} = 0. \tag{15}$$

Let

$$\sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} {}_t D_{t_2}^{\alpha+k-m} R_\nu^\alpha \frac{\mathrm{d}^{m-1-k}(\delta a^\nu)}{\mathrm{d}t^{m-1-k}} \bigg|_{t_1}^{t_2} + \sum_{\nu=1}^{2n} \sum_{k=0}^{m-1} (-1)^{m+k} \cdot {}_{t_1} D_t^{\alpha+k-m} R_\nu^\beta \frac{\mathrm{d}^{m-1-k}(\delta a^\nu)}{\mathrm{d}t^{m-1-k}} \bigg|_{t_1}^{t_2} = 0, \tag{16}$$

Equation (16) is called the transversality condition. From the condition (16), the formula (15) becomes

$$\delta A = \int_{t_1}^{t_2} \left\{ \sum_{\mu=1}^{2n} \left\{ \sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu^\alpha}{\partial a^\mu}\, {}_{t_1}^{C} D_t^\alpha a^\nu + \frac{\partial R_\nu^\beta}{\partial a^\mu}\, {}_{t}^{C} D_{t_2}^\beta a^\nu \right) - \frac{\partial B}{\partial a^\mu} + {}_t D_{t_2}^\alpha R_\mu^\alpha + {}_{t_1} D_t^\beta R_\mu^\beta \right\} \delta a^\mu \right\} \mathrm{d}t = 0.$$

According to the arbitrariness of the integral interval $[t_1, t_2]$, we obtain

$$\sum_{\mu=1}^{2n} \left\{ \sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu^\alpha}{\partial a^\mu}\, {}_{t_1}^{C} D_t^\alpha a^\nu + \frac{\partial R_\nu^\beta}{\partial a^\mu}\, {}_{t}^{C} D_{t_2}^\beta a^\nu \right) - \frac{\partial B}{\partial a^\mu} + {}_t D_{t_2}^\alpha R_\mu^\alpha + {}_{t_1} D_t^\beta R_\mu^\beta \right\} \delta a^\mu = 0. \tag{17}$$

The principle (17) is called the fractional Pfaff-Birkhoff-d'Alembert principle in terms of Caputo fractional derivative. Because of the independence of δa^μ, we obtain

$$\sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu^\alpha}{\partial a^\mu}\, {}_{t_1}^{C} D_t^\alpha a^\nu + \frac{\partial R_\nu^\beta}{\partial a^\mu}\, {}_{t}^{C} D_{t_2}^\beta a^\nu \right) - \frac{\partial B}{\partial a^\mu} + {}_t D_{t_2}^\alpha R_\mu^\alpha + {}_{t_1} D_t^\beta R_\mu^\beta = 0 (\mu = 1,\cdots,2n). \tag{18}$$

Equations (18) are called the fractional Birkhoff's equations satisfying transversality condition (16).

Now we deduce the tranditional Birkhoff's equations from the fractional Birkhoff's equations.

Let the integrand of the fractional Pfaff-Birkhoff action (9) contains only left Riemann-Liouville fractional derivatives, then the equations (18) become

$$\sum_{\nu=1}^{2n} \frac{\partial R_\nu^\alpha}{\partial a^\mu} {}_{t_1}^{C}D_t^\alpha a^\nu - \frac{\partial B}{\partial a^\mu} + {}_tD_{t_2}^\alpha R_\mu^\alpha = 0 (\mu = 1, \cdots, 2n). \quad (19)$$

When $\alpha \to 1$, the equations (19) become

$$\sum_{\nu=1}^{2n} \frac{\partial R_\nu}{\partial a^\mu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\mathrm{d}R_\mu}{\mathrm{d}t} = 0 (\mu = 1, \cdots, 2n),$$

i. e.

$$\sum_{\nu=1}^{2n} \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) \dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} \right) = 0 (\mu = 1, \cdots, 2n). \quad (20)$$

Equations (20) are the traditional Birkhoff's equations. Now the transversality condition (16) is

$$\sum_{\nu=1}^{2n} R_\nu^\alpha \delta a^\nu \Big|_{t_1}^{t_2} = 0. \quad (21)$$

From the endpoint condition (11), we find that the condition (21) is equal to zero. Hence the traditional Birkhoff's equations (20) can be obtained from the fractional Birkhoff's equations (18).

3 An illustrative example

In order to illustrate the above results, we study a Birkhoffian system whose Birkhoffian and Birkhoff's functions are

$B = (a^2)^2/2 - \ln a^1, R_1 = 0, R_2 = (a^2)^2/2 - \ln a^1$.

Try to establish the fractional Birkhoff's equations in terms Caputo fractional derivative.

Without loss of generality, we suppose that $0 < \alpha < 1, 0 < \beta < 1$. The fractional Pfaff-Birkhoff action in terms of Caputo derivative (9) gives

$$A = \int_{t_1}^{t_2} \left\{ \left[\frac{1}{2}(a^2)^2 - \ln a^1 \right] {}_{t_1}^{C}D_t^\alpha a^2 + \left[\frac{1}{2}(a^2)^2 - \ln a^1 \right] {}_t^{C}D_{t_2}^\beta a^2 - \left[\frac{1}{2}(a^2)^2 - \ln a^1 \right] \right\} \mathrm{d}t. \quad (22)$$

The fractional Pfaff-Birkhoff-d'Alembert principle in terms of Caputo fractional derivative (17) gives

$$\left(-\frac{1}{a^1} {}_{t_1}^{C}D_t^\alpha a^2 - \frac{1}{a^1} {}_t^{C}D_{t_2}^\beta a^2 + \frac{1}{a^1} \right) \delta a^1 + \left[-a^2 + {}_tD_{t_2}^\alpha \left(\frac{1}{2}(a^2)^2 - \ln a^1 \right) + {}_{t_1}D_t^\beta \left(\frac{1}{2}(a^2)^2 - \ln a^1 \right) \right] \delta a^2 = 0.$$

According to the independence of $\delta a^\nu (\nu = 1, 2)$, we obtain the fractional Birkhoff's equations (18) corresponding to the action (22) as follows

$$-\frac{1}{a^1} {}_{t_1}^{C}D_t^\alpha a^2 - \frac{1}{a^1} {}_t^{C}D_{t_2}^\beta a^2 + \frac{1}{a^1} = 0, \quad -a^2 + {}_tD_{t_2}^\alpha \left(\frac{1}{2}(a^2)^2 - \ln a^1 \right) + {}_{t_1}D_t^\beta \left(\frac{1}{2}(a^2)^2 - \ln a^1 \right) = 0,$$

and the transversality condition (16) gives

$${}_tD_{t_2}^{\alpha-1} \left(\frac{1}{2}(a^2)^2 - \ln a^1 \right) \delta a^2 \Big|_{t_1}^{t_2} - {}_{t_1}D_t^\beta \left(\frac{1}{2}(a^2)^2 - \ln a^1 \right) \delta a^2 \Big|_{t_1}^{t_2} = 0.$$

4 Conclusion

The natural world is fractional essentially, using the fractional mathematical models can overcome the shortcomings that the theorem established by the classical mathematical models is not coincide well with the experimental results, and comparing with the nonlinear models, the physical meaning of the fractional models is more distinct. Moreover, the fractional calculus is having the global relevance, so it can reflect the historical dependent process of the systematic functions' development. The study of the fractional Birkhoffian system is an advanced task, there are many problems to be solved in it. In this paper, the fractional Pfaff-Birkhoff principle and the Birkhoff's equations are studied, the fractional Birkhoff's equations are established, and the corresponding transversality condition is given, in addition, the integer order derivative is the special case of this paper.

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基于 Caputo 导数的分数阶 Pfaff-Birkhoff 原理和 Birkhoff 方程

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摘要: 研究了在 Caputo 分数阶导数下的分数阶 Pfaff-Birkhoff 变分问题. 首先给出了 Caputo 分数阶导数的定义,以及相应的分部积分公式和交换关系,其次建立了分数阶 Pfaff-Birkhoff 原理和分数阶 Birkhoff 方程,最后举例说明结果的应用.

关键词: 分数阶 Pfaff-Birkhoff 原理; 分数阶 Birkhoff 方程; Caputo 分数阶导数; 横截性条件

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