第38卷 第3期 江西师范大学学报(自然科学版) Vol. 38 No. 3
2014年5月 Journal of Jiangxi Normal University(Natural Science) May 2014

文章编号:1000-5862(2014)03-0258-03

3维Boussinesq方程组正则性准则的一个注记

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摘要: 利用奇异积分理论和广义能量不等式研究3维不可压缩Boussinesq方程组,得到了该方程组的1个正则性准则,推广了已有的结论.

关键词: Boussinesq方程组;正则性准则;Morrey-Campanato空间

中图分类号: O 175.2 **文献标志码:** A

0 引言

本文考虑3维Boussinesq方程组的柯西问题:

$$\begin{cases}\dfrac{\partial u}{\partial t}+(u\cdot\nabla)u-\mu\Delta u+\nabla P=\theta e_3,\\ (x,t)\in\mathbf{R}^3\times(0,\infty),\\ \dfrac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta-\kappa\Delta\theta=0,\\ \nabla\cdot u=0,\\ u(x,0)=u_0,\theta(x,0)=\theta_0,\end{cases}\tag{1}$$

其中$u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$表示流体速度,$P=P(x,t)$为压力,$\theta=\theta(x,t)$为温度,$\mu$为粘性系数,$\kappa$为热扩散系数,$e_3=(0,0,1)^{\mathrm{T}}$,$\mu_0$与$\theta_0$分别为在$t=0$时给定的流体初始速度与初始温度,且满足$\nabla\cdot u_0=0$.

Boussinesq方程组不仅在大气科学中有着重要应用[1],而且在地球物理科学中也得到了广泛应用[2]. J. R. Cannon等[3]得到了在2维情况下方程组(1)关于整体时间的正则性解. 然而,当$\mu=\kappa=0$时,该方程解的正则性问题仍然是数学流体力学的公开问题[4-8]. 而对于无粘性或者无扩散情形下的解的整体正则性问题,相关研究可参见文献[9-14]. 对于3维Boussinesq方程组(1),文献[15]给出了如下光滑解的正则性准则:

$$\nabla u\in L^1(0,T;L^\infty(\mathbf{R}^3)).$$

最近,Qiu Hua等得到3维Boussinesq方程组的Serrin类正则性准则[16].

本文研究3维Boussinesq方程组的正则性问题,得到该方程组的1个正则性准则,主要结论为

定理1 设流体的初始速度与温度$(u_0,\theta_0)\in H^2(\mathbf{R}^3)$,且$(u,\theta)$为问题(1)在$0\leqslant t\leqslant T$时的光滑解,若

$$\int_0^T\left(1+\|u(t)\|_{\dot M_{2,3/r}}^{2/(1-r)}\right)\mathrm{d}t<\infty,\tag{2}$$

则解(u,θ)在时刻$t=T$处仍然是光滑的,其中$0<r<1$.

1 预备知识

本节给出Morrey-Campanato空间的相关定义以及证明中需要的引理.

定义1 设$0<p\leqslant q\leqslant\infty$,Morrey-Campanato空间$\dot M_{p,q}$定义为

$$\dot M_{p,q}=\left\{f\in L^p_{loc}(\mathbf{R}^3)\ \middle|\ \|f\|_{\dot M_{p,q}}=\sup_{x\in\mathbf{R}^3}\sup_{R>0}R^{3/q-3/p}\|f\|_{L^p(B(x,R))}<\infty\right\},$$

其中$B(x,R)$是$\mathbf{R}^3$中以x为中心且半径为R的球.

对于Morrey-Campanato空间与Lorentz空间,当$p\geqslant 2$时,有如下嵌入关系:

$$L^{3/r}(\mathbf{R}^3)\subset L^{3/r,\infty}(\mathbf{R}^3)\subset\dot M_{p,3/r}(\mathbf{R}^3)\subset\dot X(\mathbf{R}^3)\subset\dot M_{2,3/r}(\mathbf{R}^3).$$

收稿日期:2014-03-20

基金项目:国家自然科学基金(11126266,11271141)和广东高校优秀青年创新人才培养计划(LYM11030,2012LYM_0030)资助项目.

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对于空间 $\dot{M}_{p,q}$ 和空间 $\dot{N}_{p',q'}$,有如下引理.

引理 1[17]　设 $0 \leqslant r < 3/2$,且空间 $\dot{Z}_r$ 定义为如下函数 f 的集合:

$$f(x) \in L^2_{loc}(\mathbf{R}^3)$$

且有

$$\| f \|_{\dot{Z}_r} = \sup_{\| g \|_{\dot{B}^r_{2,1}} \leqslant 1} \| fg \|_{L^2} < \infty,$$

则 $f \in \dot{M}_{2,3/r}(\mathbf{R}^3)$ 当且仅当 $f \in \dot{Z}_r$.

进一步地,有如下引理.

引理 2[18]　设 $0 < r < 1$,则有

$$\| f \|_{\dot{B}^r_{2,1}} \leqslant C \| f \|^{1-r}_{L^2} \| \nabla f \|^r_{L^2}.$$

2　定理 1 的证明

不失一般性,为方便表述,假设 $\mu = \kappa = 1$. 首先,(1) 式的第 1 个方程与第 2 个方程两端分别乘以 Δu 与 $\Delta\theta$,并将所得方程在 $\mathbf{R}^3$ 上积分,得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \nabla u \|^2_{L^2} + \| \Delta u \|^2_{L^2} = \int [(u \cdot \nabla) u] \cdot \Delta u \mathrm{d}x + \int \nabla P \cdot \Delta u \mathrm{d}x - \int (\theta \boldsymbol{e}_3) \cdot \Delta u \mathrm{d}x = -\int \partial_i u_j \partial_k u_i \partial_k u_j \mathrm{d}x - \int \partial_k(\theta \boldsymbol{e}_3) \partial_k u_j \mathrm{d}x, \tag{3}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| \nabla \theta \|^2_{L^2} + \| \Delta \theta \|^2_{L^2} = \int [(u \cdot \nabla) \theta] \cdot \Delta \theta \mathrm{d}x = -\int \partial_k u_i \partial_i \theta \partial_k \theta \mathrm{d}x. \tag{4}$$

这里用到了不可压缩条件 $\nabla \cdot u = 0$.

将(3) 式与(4) 式相加,得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2}) + \| \Delta u \|^2_{L^2} + \| \Delta \theta \|^2_{L^2} = -\int \partial_i u_j \partial_k u_i \partial_k u_j \mathrm{d}x - \int \partial_k(\theta \boldsymbol{e}_3) \partial_k u_j \mathrm{d}x - \int \partial_k u_i \partial_i \theta \partial_k \theta \mathrm{d}x.$$

对上式右边分部积分,得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2}) + \| \Delta u \|^2_{L^2} + \| \Delta \theta \|^2_{L^2} = -\int u_i \partial_k(\partial_i u_j \partial_k u_j) \mathrm{d}x + \int u_i \partial_k(\partial_i \theta \partial_k \theta) \mathrm{d}x + \int \partial_k(\theta \boldsymbol{e}_3) \mathrm{d}x \triangleq I + II + III. \tag{5}$$

根据引理 1、引理 2、Hölder 不等式以及 Young 不等式,得

$$\begin{aligned} |I| &\leqslant C \| u \|_{\dot{M}_{2,3/r}} \| \nabla u \|_{\dot{B}^r_{2,1}} \| \Delta u \|^2_{L^2} \leqslant \\ &C \| u \|_{\dot{M}_{2,3/r}} \| \nabla u \|^{1-r}_{L^2} \| \Delta u \|^{1+r}_{L^2} \leqslant \\ &C \| u \|^{2/(1-r)}_{\dot{M}_{2,3/r}} \| \nabla u \|^2_{L^2} + \frac{1}{2} \| \Delta u \|^2_{L^2}. \end{aligned}$$

类似地,可得

$$|II| \leqslant C \| u \|^{2/(1-r)}_{\dot{M}_{2,3/r}} \| \nabla \theta \|^2_{L^2} + \frac{1}{2} \| \Delta \theta \|^2_{L^2},$$

$$|III| \leqslant C(\| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2}).$$

将上面关于 I,II,III 的估计代入(5) 式,得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2}) + \| \Delta u \|^2_{L^2} + \| \Delta \theta \|^2_{L^2} \leqslant C(1 + \| u \|^{2/(1-r)}_{\dot{M}_{2,3/r}})(\| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2}).$$

对上式应用 Gronwall 不等式,可以得到在假设(2) 下解(u,θ) 在时刻 $t = T$ 处仍然是光滑的. 定理 1 证毕.

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A Note on Regularity Criterion for 3D Boussinesq Equations

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Abstract: The three-dimensional Boussinesq equations with the incompressibility condition is considered by the singular integrals theory and the generalized energy inequality. And one regularity criterion for the 3D Boussinesq equations is obtained,which extend the known results.

Key words: Boussinesq equations; regularity criterion; Morrey-Campanato spaces

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The Growth for Solutions of a Class of Higher Order Linear Differential Equations with Meromorphic Coefficients

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Abstract: The growth of solutions of the differential equation $f^{(k)}+\cdots+A_0f=0(k\geqslant 2)$ is investigated by using the fundamental theory of Nevanlinna value distribution,where $A_j(0\leqslant j\leqslant k-1)$ are meromorphic functions. It is proved that every nontrivial solution f of the equation is of infinite order with giving some different condition on A_j $(0\leqslant j\leqslant k-1)$.

Key words: differential equation; meromorphic function; deficient value; infinite order

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