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一类高阶线性微分方程解在角域上的增长性

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摘要:主要研究了高阶微分方程 $f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=0$ 的解在角域上的增长性,其中 A_0,A_j $(1\leqslant j\leqslant k-1)$ 为亚纯函数,且假设 A_0 以有限复数 a 为亏值,$\rho(A_j)=0(1\leqslant j\leqslant k-1)$,通过给定适当的条件,证明了齐次线性微分方程的任一非零解在某些角域上的增长级为无穷.

关键词:微分方程;亚纯函数;亏值;角域上的增长级

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0 引言与结果

本文使用 Nevanlinna 值分布理论的标准记号[1-2],以 $T(r,f)$ 记亚纯函数 $f(z)$ 的特征函数,$N(r,f=a)$ 表示函数 $f(z)$ 的 a 值点的密值量,$\rho(f)$ 和 $\mu(f)$ 分别表示 $f(z)$ 的级与下级,用 $\delta(a,f)$ 表示函数 $f(z)$ 在点 a 的亏量等.

关于亚纯系数高阶线性微分方程

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=0 \quad (1)$$

非零解在全平面上的增长性已有很多研究,本文主要讨论此类方程的解在某些角域上的增长性.

论文的证明需要用到角域上特征函数的相关性质[3-4].

假设 $f(z)$ 是角域 $\Omega(\alpha,\beta)$ 上的亚纯函数,其中 $0<\beta-\alpha\leqslant 2\pi$,$\omega=\pi/(\beta-\alpha)$. 记 $A_{\alpha\beta}(r,f)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\right\}\frac{dt}{t}$;

$$B_{\alpha\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)\,d\theta;$$

$$C_{\alpha\beta}(r,f)=2\sum_{1<|b_v|<r}\left(\frac{1}{|b_v|^\omega}-\frac{|b_v|^\omega}{r^{2\omega}}\right)\sin\omega(\beta_v-\alpha);$$

$$D_{\alpha\beta}(r,f)=A_{\alpha\beta}(r,f)+B_{\alpha\beta}(r,f);$$

$$S_{\alpha\beta}(r,f)=A_{\alpha\beta}(r,f)+B_{\alpha\beta}(r,f)+C_{\alpha\beta}(r,f),$$

其中 $b_v=|b_v|e^{i\beta_v}(v=1,2,\cdots)$ 为 $f(z)$ 在角域 $\Omega(\alpha,\beta)$ 内的极点,并且重级极点按重数计算.

亚纯函数 $f(z)$ 在角域 $\Omega(\alpha,\beta)$ 上的增长级和下级依次定义为

$$\sigma_{\alpha\beta}(f)=\limsup_{r\to\infty}\log^+S_{\alpha\beta}(r,f)/\log r,$$

$$\mu_{\alpha\beta}(f)=\liminf_{r\to\infty}\log^+S_{\alpha\beta}(r,f)/\log r.$$

下面定义角域上的 Ahlfors-Shimizu 特征函数[5],令

$$\Omega(r)=\{z:\alpha<\arg z<\beta,0<|z|<r\},$$

定义

$$S(r,\Omega,f)=\frac{1}{\pi}\iint_{\Omega(r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2d\sigma,$$

$$T_0(r,\Omega,f)=\int_1^r\frac{S(t,\Omega,f)}{t}dt.$$

并运用 Ahlfors-Shimizu 特征函数定义了 $f(z)$ 在角域 $\Omega(\alpha,\beta)$ 上的级和下级:

$$\overline{\sigma}_{\alpha\beta}(f)=\limsup_{r\to\infty}\log^+T_0(r,\Omega,f)/\log r,$$

$$\overline{\mu}_{\alpha\beta}(f)=\liminf_{r\to\infty}\log^+T_0(r,\Omega,f)/\log r.$$

然而这 2 种不同定义的增长级存在一定的联系,根据文献[6]中证明的不等式

$$S_{\alpha\beta}(r,f)\leqslant 2\omega^2\frac{T_0(r,\Omega,f)}{r^\omega}+\omega^3\int_1^r\frac{T_0(t,\Omega,f)}{t^{\omega+1}}dt+O(1). \quad (2)$$

其中 $\Omega=\Omega(\alpha,\beta)$,$\omega=\pi/(\beta-\alpha)$,容易知道,若 $\overline{\sigma}_{\alpha\beta}(f)<\infty$. 则 $\sigma_{\alpha\beta}(f)<\infty$.

若 $\{\alpha_j,\beta_j\}(j=1,2,\cdots,q)$ 为 q 组实数,满足

$$-\pi\leqslant\alpha_1<\beta_1\leqslant\alpha_2<\beta_2\cdots\leqslant\alpha_q<\beta_q<\pi,\quad \alpha_{q+1}=\alpha_1+2\pi. \quad (3)$$

记 $X=\bigcup_{j=1}^q\{z:\alpha_j\leqslant\arg z\leqslant\beta_j\}$. 定义 C 平面上的亚纯函数 $f(z)$ 在角域并 $X=\bigcup_{j=1}^q\{z:\alpha_j\leqslant\arg z\leqslant\beta_j\}$

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上的级为

$$\overline{\sigma}_X(f) = \limsup_{r\to\infty} \log^+ T_0(r,X,f)/\log r.$$

显然$\overline{\sigma}_{\alpha_j\beta_j}(f) \leqslant \overline{\sigma}_X(f) \leqslant \sum_{j=1}^{q} \overline{\sigma}_{\alpha_j\beta_j}(f)$,$j=1,2,\cdots,q$. 如果$\overline{\sigma}_X(f) = +\infty$,则至少存在1个$j_0(1\leqslant j_0\leqslant q)$,使得$\overline{\sigma}_{\alpha_{j0}\beta_{j0}}(f) = +\infty$.

定理A[7] 设$A_0(z)$于开平面亚纯,具有非零的增长级$\rho(0<\rho\leqslant\infty)$,其下级$\mu=\mu(A_0)<\infty$,在$\infty$的亏量$\delta=\delta(\infty,A_0)>0$. $\{\alpha_j,\beta_j\}(j=1,2,\cdots,q)$为满足(3)式的$q$组实数,并且$\sum_{j=1}^{q}(\alpha_{j+1}-\beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\delta/2}$,其中$\sigma>0$,$\mu\leqslant\sigma\leqslant\rho$. 若$A_j(z)(j=1,2,\cdots,k)$于开平面亚纯且$T(r,A_j)=o(T(r,A_0))$,则方程

$$A_kf^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=0$$

的每1个非零解$f(z)$有$\overline{\sigma}_X(f)=+\infty$,其中$X=\bigcup_{j=1}^{q}\{z:\alpha_j\leqslant\arg z\leqslant\beta_j\}$.

自然会问:当$A_0(z)$为具有有限亏值的亚纯函数时,方程(1)是否也有相同的结论?下面的定理回答了上述问题.

定理1 设$A_0(z)$为$\rho=\rho(A_0)$级亚纯函数,其下级$\mu=\mu(A_0)<\infty$,极点收敛指数$\lambda=\lambda(1/A_0)<\rho$,存在有限亏值$a$,$\delta=\delta(a,A_0)>0$. $\{\alpha_j,\beta_j\}(j=1,2,\cdots,q)$为满足(3)式的$q$组实数,令$\omega=\max\{\pi/(\beta_1-\alpha_1),\cdots,\pi/(\beta_q-\alpha_q)\}$,且$\omega<\rho$,$\sum_{j=1}^{q}(\alpha_{j+1}-\beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\delta/2}$,其中$\sigma>0$,$\max\{\omega,\mu,\lambda\}<\sigma\leqslant\rho$. 若$A_j(z)(j=1,2,\cdots,k-1)$于开平面亚纯且$\rho(A_j)=0$,则方程

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=0$$

的所有非零解$f(z)$有$\overline{\sigma}_X(f)=+\infty$,其中$X=\bigcup_{j=1}^{q}\{z:\alpha_j\leqslant\arg z\leqslant\beta_j\}$.

1 引理

为了证明定理,还需要下面的相关记号和引理.

定义1 假设$B(z_n,r_n)=\{z:|z-z_n|<r_n\}$为复平面上的1列开圆域,如果$\sum_{n=1}^{\infty}r_n<\infty$,则称$\bigcup_{n=1}^{\infty}B(z_n,r_n)$为1个$R$-集.

引理1[8] 设$f(z)$为角域$\overline{\Omega}(\alpha,\beta)$上的有限$\rho$级亚纯函数,令$\Gamma=\{(n_1,m_1),(n_2,m_2),\cdots,(n_j,m_j)\}$表示满足$n_i>m_i\geqslant0(i=1,2,\cdots,j)$的不同整数对的有限集合,设$\varepsilon>0$及$\delta>0$为给定的正常数,则存在只与$f,\varepsilon,\delta$有关的常数$k>0$,使得

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| < k|z|^{(n-m)(k_\delta+2\rho+1+\varepsilon)}(\sin k_\delta(\varphi-\alpha-\delta))^{-2(n-m)}$$

对所有$(n,m)\in\Gamma$成立,其中$z=re^{i\varphi}\in\Omega(\alpha+\delta,\beta-\delta)$且$z\notin D$,$D$为由半径之和为有限的可数个圆盘构成的$R$-集,其中$k_\delta=\pi/(\beta-\alpha-2\delta)$.

引理2[6,9] 设$f(z)$为开平面上的超越亚纯函数,下级μ和级ρ分别满足$\mu<\infty$和$0<\rho\leqslant\infty$,则对任意满足$\mu\leqslant\sigma\leqslant\rho$的正数$\sigma$和1个线性测度为有限的集合$E$,存在1个正数序列$\{r_n\}$满足以下条件:

(i) $r_n\notin E$,$\lim\limits_{n\to\infty}r_n/n=\infty$;

(ii) $\liminf\limits_{n\to\infty}\log T(r_n,f)/\log r_n\geqslant\sigma$;

(iii) $T(t,f)\leqslant(1+o(1))(2t/r_n)^{\sigma}T(r_n/2,f)$,$t\in[r_n/n,nr_n]$,

并称满足上述条件的序列$\{r_n\}$为$f(z)$在除集E外的σ级Pólya峰序列.

设$f(z)$为开平面上的亚纯函数,a为复常数,令

$$D(r,a)=\left\{\theta\in[-\pi,\pi):\log^+\frac{1}{|f(re^{i\theta})-a|}>\frac{1}{\log r}T(r,f)\right\}.$$

引理3[10] 设$f(z)$为开平面上的超越亚纯函数,下级μ和级ρ分别满足$\mu<\infty$和$0<\rho\leqslant\infty$,若$f(z)$具有亏值$a\in\hat{c}=c\cup\{\infty\}$,$\delta=\delta(a,f)>0$,则对$f(z)$的任一$\sigma(\mu\leqslant\sigma\leqslant\rho)$级Pólya峰序列$\{r_n\}$有

$$\liminf_{n\to\infty} m_{eas}D(r_n,a)\geqslant \min\{2\pi,\frac{4}{\sigma}\arcsin\sqrt{\delta(a,f)/2}\}.$$

引理4[11] 设$f(z)$为开平面上的亚纯函数,$\Omega(\alpha,\beta)$为开平面上的任一角域,$0<\beta-\alpha\leqslant2\pi$,则对任意复数$a\neq\infty$,有$S_{\alpha\beta}(r,1/(f-a))=S_{\alpha\beta}(r,f)+\varepsilon(r,a)$,其中$\varepsilon(r,a)=O(1)(r\to\infty)$.

引理5[12] 设$f(z)$为开平面上的亚纯函数,当$f(0)\neq\infty$时,

$$|T(r,f)-T_0(r,f)-\log^+|f(0)||\leqslant\frac{1}{2}\log2,$$

当$f(0)=\infty$时,

$$|T(r,f)-T_0(r,f)-\log|C_{-m}||\leqslant\frac{1}{2}\log2,$$

其中C_{-m}表示$f(z)$在$z=0$点邻域内展式中的首项

非零系数.

2 定理的证明

若存在非零解 $f(z)$ 使得 $\overline{\sigma}_X(f) < +\infty$,则有 $\overline{\sigma}_{\alpha_j\beta_j}(f) \leqslant \overline{\sigma}_X(f) < +\infty$,$j=1,2,\cdots,q$.

由方程(1)有

$$|A_0(z)| \leqslant \left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=1}^{k-1}\left|\frac{f^{(j)}(z)}{f(z)}\right||A_j(z)|.$$

(i) 若 $\max\{\omega,\mu,\lambda\} < \sigma < \rho$.

由不等式 $\sum_{j=1}^{q}(\alpha_{j+1}-\beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\delta/2}$ 及 $\omega=\max\{\frac{\pi}{\beta_1-\alpha_1},\cdots,\frac{\pi}{\beta_q-\alpha_q}\} < \sigma$ 知 $\exists\varepsilon>0$,使得

$$\sum_{j=1}^{q}(\alpha_{j+1}-\beta_j+2\varepsilon)+2\varepsilon < \frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\delta/2},\quad(4)$$

$\omega_\varepsilon=\max\left\{\frac{\pi}{\beta_1-\alpha_1-\frac{4}{3}\varepsilon},\cdots,\frac{\pi}{\beta_q-\alpha_q-\frac{4}{3}\varepsilon}\right\}<\sigma$ 及 $\max\{\omega,\mu,\lambda\} < \sigma+2\varepsilon \leqslant \rho$ 同时成立.

由引理1可知,对上述 ε 和对给定的 $\varepsilon_0>0$ 以及充分大的 N,对每个 j 存在 η_j 满足

$$(\beta_j-\alpha_j)/N < \eta_j < \varepsilon/6$$

和只与 f,ε_0,η_j 有关的常数 $h>0$,使得当 $z=re^{i\varphi}\in\Omega(\alpha_j+\eta_j,\beta_j-\eta_j)$,$z\notin D$ 时,对所以的 $(n,m)\in\Gamma$ 有

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| < h\,|z|^{(n-m)(k_\delta+2\rho+1+\varepsilon)}(\sin k_\delta(\varphi-\alpha_j-\eta_j))^{-2(n-m)},$$

其中 $k_\delta=\pi/(\beta_j-\alpha_j-2\eta_j)$,$D$ 为引理1给出的 R-集.

特别地,当 $z=re^{i\varphi}\in\Omega(\alpha_j+2\eta_j,\beta_j-2\eta_j)$ 时上式成立,且 $\varphi-\alpha_j-\eta_j>\eta_j>0$,此时 $|\sin k_\delta(\varphi-\alpha_j-\eta_j)|>l$($l$ 为某一常数),所以当 $z=re^{i\varphi}\in\Omega(\alpha_j+2\eta_j,\beta_j-2\eta_j)$,$z\notin D$ 时,$\exists M>0$,不依赖 z,使得

$$|f^{(j)}(z)/f(z)| < |z|^M(j=1,2,\cdots,k).\quad(5)$$

令 $F=\{r|z\in D,\text{s.t}\,|z|=r\}$,则 $m_{eas}F<\infty$.

对 $A_0(z)$ 应用引理2,$\exists\sigma+2\varepsilon$ 级 Pólya 峰序列 $\{r_n\}$,使

$$\liminf_{n\to\infty}\frac{\log T(r_n,A_0)}{\log r_n}\geqslant\sigma+2\varepsilon,\quad(6)$$

其中 $r_n\notin F$.

由引理3,对充分大的 n 有

$$m_{eas}D(r_n,a)\geqslant\min\{2\pi,\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\delta/2}\}.$$

但由 $0<\delta<1$,$1/2\leqslant\omega<\sigma$ 可知,

$$\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\delta/2}<2\pi,$$

则可得

$$m_{eas}D(r_n,a)>\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\delta/2}-\varepsilon.\quad(7)$$

令 $K_n=m_{eas}(D(r_n,a)\cap\bigcup_{j=1}^{q}(\alpha_j+\varepsilon,\beta_j-\varepsilon))$,则由(4)式和(7)式可得

$$K_n\geqslant m_{eas}(D(r_n,a))-m_{eas}([0,2\pi)\setminus\bigcup_{j=1}^{q}(\alpha_j+\varepsilon,\beta_j-\varepsilon))=m_{eas}(D(r_n,a))-m_{eas}(\bigcup_{j=1}^{q}(\beta_j-\varepsilon,\alpha_{j+1}+\varepsilon))=m_{eas}(D(r_n,a))-\sum_{j=1}^{q}(\alpha_{j+1}-\beta_j+2\varepsilon)>\varepsilon>0.$$

令 $K=\liminf_{n\to\infty}K_n$,则 $0<\varepsilon\leqslant K$. 从而 $\exists j_0$ 使得有无限个 n 满足

$$m_{eas}(D(r_n,a)\cap(\alpha_{j_0}+\varepsilon,\beta_{j_0}-\varepsilon))>K/q.$$

不妨设对所有 n 都成立,若不然只须取其子列代替. 当 n 充分大时,有

$$m_{eas}D'(r_n)=m_{eas}\{\theta:r_ne^{i\theta}\in D\}=0.$$

令 $D_n=(\alpha_{j_0}+\varepsilon,\beta_{j_0}-\varepsilon)\setminus D'(r_n)$,$E_n=D(r_n,a)\cap D_n$,由 $D(r_n,a)$ 的定义可知

$$\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|A_0(r_ne^{i\theta})-a|}d\theta\geqslant\int_{E_n}\log^+\frac{1}{|A_0(r_ne^{i\theta})-a|}d\theta\geqslant m_{eas}E_n\cdot\frac{T(r_n,A_0)}{\log r_n}>\frac{K}{q}\cdot\frac{T(r_n,A_0)}{\log r_n}.\quad(8)$$

又设 B_1:$\arg z=\alpha_{j_0}+\varepsilon'$,$B_2$:$\arg z=\beta_{j_0}-\varepsilon'$ 为由原点发出的2条半直线,当 r_n 充分大时,取 $\varepsilon'\in(\varepsilon/3,2\varepsilon/3)$ 使得 B_1,B_2 只与 D 中的有限个圆盘相交,令 $M_1=\{r|re^{i\theta}\in B_1\cap D\}$,$M_2=\{r|re^{i\theta}\in B_2\cap D\}$,则 $m_{eas}M_1<+\infty$,$m_{eas}M_2<+\infty$.

由 $B_{\alpha\beta}(r,f)$ 的定义可得

$$B_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{1}{A_0-a}\right)=\frac{2\omega_{j_0}}{\pi r_n^{\omega_{j_0}}}\int_{\alpha_{j_0}+\varepsilon'}^{\beta_{j_0}-\varepsilon'}\log^+\frac{1}{|A_0(r_ne^{i\theta})-a|}\sin(\omega_{j_0}(\theta-\alpha_{j_0}-\varepsilon'))d\theta\geqslant\frac{2\omega_{j_0}}{\pi r_n^{\omega_{j_0}}}\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|A_0(r_ne^{i\theta})-a|}\cdot\sin(\omega_{j_0}(\theta-\alpha_{j_0}-\varepsilon'))d\theta\geqslant\frac{2\omega_{j_0}}{\pi r_n^{\omega_{j_0}}}\sin(\omega_{j_0}(\varepsilon-\varepsilon'))\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|A_0(r_ne^{i\theta})-a|}d\theta,$$

即得

$$\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|A_0(r_n e^{i\theta})-a|}d\theta\leqslant$$

$$\frac{\pi r_n^{\omega_{j_0}}}{2\omega_{j_0}\sin(\omega_{j_0}(\varepsilon-\varepsilon'))}B_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{1}{A_0-a}\right),\quad(9)$$

其中 $\omega_{j_0}=\pi/(\beta_{j_0}-\alpha_{j_0}-2\varepsilon')$.

又由引理 4 可得

$$B_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{1}{A_0-a}\right)\leqslant$$

$$S_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{1}{A_0-a}\right)=S_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_0)+$$

$$O(1)=D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_0)+$$

$$C_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_0)+O(1).\quad(10)$$

由 $r_n\notin F$ 及 B_1,B_2 只与 D 中的有限个圆盘相交可得

$$D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{f^{(j)}}{f}\right)=A_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{f^{(j)}}{f}\right)+$$

$$B_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{f^{(j)}}{f}\right)=\frac{\omega_{j_0}}{\pi}\int_1^{r_n}\left(\frac{1}{t^{\omega_{j_0}}}-\frac{t^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\cdot$$

$$\left\{\log^+\left|\frac{f^{(j)}(te^{i(\alpha_{j_0}+\varepsilon')})}{f(te^{i(\alpha_{j_0}+\varepsilon')})}\right|+\log^+\left|\frac{f^{(j)}(te^{i(\beta_{j_0}-\varepsilon')})}{f(te^{i(\beta_{j_0}-\varepsilon')})}\right|\right\}\frac{dt}{t}+$$

$$\frac{2\omega_{j_0}}{\pi r_n^{\omega_{j_0}}}\int_{\alpha_{j_0}+\varepsilon'}^{\beta_{j_0}-\varepsilon'}\log^+\left|\frac{f^{(j)}(r_ne^{i\theta})}{f(r_ne^{i\theta})}\right|\sin(\omega_{j_0}(\theta-\alpha_{j_0}-\varepsilon'))d\theta.$$

又由于

$$\frac{\omega_{j_0}}{\pi}\int_1^{r_n}\left(\frac{1}{t^{\omega_{j_0}}}-\frac{t^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\log^+\left|\frac{f^{(j)}(te^{i(\alpha_{j_0}+\varepsilon')})}{f(te^{i(\alpha_{j_0}+\varepsilon')})}\right|\frac{dt}{t}=$$

$$\frac{\omega_{j_0}}{\pi}\int_{M_1}\left(\frac{1}{t^{\omega_{j_0}}}-\frac{t^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\log^+\left|\frac{f^{(j)}(te^{i(\alpha_{j_0}+\varepsilon')})}{f(te^{i(\alpha_{j_0}+\varepsilon')})}\right|\frac{dt}{t}+$$

$$\frac{\omega_{j_0}}{\pi}\int_{[1,r_n]\setminus M_1}\left(\frac{1}{t^{\omega_{j_0}}}-\frac{t^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\log^+\left|\frac{f^{(j)}(te^{i(\alpha_{j_0}+\varepsilon')})}{f(te^{i(\alpha_{j_0}+\varepsilon')})}\right|\frac{dt}{t}.$$

由(5) 式,令 $z=re^{i(\alpha_{j_0}+\varepsilon')}$,当 $r\in[1,r_n]\setminus M_1$ 时,有

$$|f^{(j)}(z)/f(z)|<|z|^M.$$

即可得

$$\frac{\omega_{j_0}}{\pi}\int_1^{r_n}\left(\frac{1}{t^{\omega_{j_0}}}-\frac{t^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\log^+\left|\frac{f^{(j)}(te^{i(\alpha_{j_0}+\varepsilon')})}{f(te^{i(\alpha_{j_0}+\varepsilon')})}\right|\frac{dt}{t}=O(1).$$

同理可得

$$\frac{\omega_{j_0}}{\pi}\int_1^{r_n}\left(\frac{1}{t^{\omega_{j_0}}}-\frac{t^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\log^+\left|\frac{f^{(j)}(te^{i(\beta_{j_0}-\varepsilon')})}{f(te^{i(\beta_{j_0}-\varepsilon')})}\right|\frac{dt}{t}=O(1),$$

即得

$$A_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,f^{(j)}/f)=O(1).$$

又由 $z=r_ne^{i\theta}\in\Omega(\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon')\subseteq\Omega(\alpha_j+2\eta_j,\beta_j-2\eta_j)$ 可知,有 $|f^{(j)}(z)/f(z)|<|z|^M$,则可得 $B_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,f^{(j)}/f)=O(1)$,即得

$$D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,f^{(j)}/f)=O(1)\ (1\leqslant j\leqslant k).$$

对 $A_j(z)\ (j=1,2,\cdots,k)$ 应用不等式(2)

$$S_{\alpha\beta}(r,f)\leqslant 2\omega^2\frac{T_0(r,\Omega,f)}{r^\omega}+\omega^3\int_1^r\frac{T_0(t,\Omega,f)}{t^{\omega+1}}dt+O(1),$$

其中 $\Omega=\Omega(\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon')$,$\omega=\pi/(\beta_{j_0}-\alpha_{j_0}-2\varepsilon')$,由引理 5 可知 $T_0(r,f)=T(r,f)+O(1)$,则 $D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_j)\leqslant S_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_j)=O(T_0(r_n,\Omega,A_j))=O(T_0(r_n,A_j))=O(T(r_n,A_j))$.

从而

$$D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_0)\leqslant\sum_{j=1}^{k}D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,f^{(j)}/f)+$$

$$\sum_{j=1}^{k-1}D_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_j)=O\left(\sum_{j=1}^{k-1}T(r_n,A_j)\right)+O(1).\quad(11)$$

设 $b_v=|b_v|e^{i\beta_v}\ (v=1,2,\cdots)$ 为 $A_0(z)$ 在角域 $\Omega(\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon')$ 内的所有极点,重级极点按重数计算,由定义可得

$$C_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_0)=2\sum_{1<|b_v|<r_n}\left(\frac{1}{|b_v|^{\omega_{j_0}}}-\right.$$

$$\left.\frac{|b_v|^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)\sin\omega_{j_0}(\beta_v-\alpha_{j_0}-\varepsilon')\leqslant 2\sum_{1<|b_v|<r_n}\left(\frac{1}{|b_v|^{\omega_{j_0}}}-\right.$$

$$\left.\frac{|b_v|^{\omega_{j_0}}}{r_n^{2\omega_{j_0}}}\right)=2\int_1^{r_n}\frac{dn(t)}{t^{\omega_{j_0}}}-\frac{2}{r_n^{2\omega_{j_0}}}\int_1^{r_n}t^{\omega_{j_0}}dn(t)=$$

$$2\left(\frac{n(t)}{t^{\omega_{j_0}}}\Big|_1^{r_n}+\omega_{j_0}\int_1^{r_n}\frac{n(t)}{t^{1+\omega_{j_0}}}dt\right)-\frac{2}{r_n^{2\omega_{j_0}}}\left(t^{\omega_{j_0}}n(t)\Big|_1^{r_n}-\right.$$

$$\left.\omega_{j_0}\int_1^{r_n}t^{\omega_{j_0}-1}n(t)dt\right).$$

由 $\limsup\limits_{n\to\infty}\log n(t)/\log t=\lambda$,可取 $\varepsilon''>0$ 使得 $\lambda+\varepsilon''<\sigma$,当 n 充分大时,有 $n(t)<t^{\lambda+\varepsilon''}$,故有

$$\int_1^{r_n}\frac{n(t)}{t^{\omega_{j_0}+1}}dt\leqslant\int_1^{r_n}t^{\lambda-\omega_{j_0}-1+\varepsilon''}dt<\frac{1}{\lambda-\omega_{j_0}+\varepsilon''}r_n^{\lambda-\omega_{j_0}+\varepsilon''}.$$

则

$$C_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}(r_n,A_0)\leqslant\frac{2n(r_n)}{r_n^{\omega_{j_0}}}+\frac{\omega_{j_0}}{\lambda-\omega_{j_0}+\varepsilon''}\cdot$$

$$r_n^{\lambda-\omega_{j_0}+\varepsilon''}-\frac{2n(r_n)}{r_n^{\omega_{j_0}}}+2\frac{n(r_n)}{r_n^{\omega_{j_0}}}+O(1)=$$

$$O(r_n^{\lambda-\omega_{j_0}+\varepsilon''})+O(1).\quad(12)$$

将(11) ~ (12) 代入(10) 式,再由(9) 式可得

$$\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon}\log^+\frac{1}{|A_0(r_ne^{i\theta})-a|}d\theta\leqslant$$

$$\frac{\pi r_n^{\omega_{j_0}}}{2\omega_{j_0}\sin(\omega_{j_0}(\varepsilon-\varepsilon'))}B_{\alpha_{j_0}+\varepsilon',\beta_{j_0}-\varepsilon'}\left(r_n,\frac{1}{A_0-a}\right)=$$

$$r_n^{\omega_{j_0}}\left(O\left(\sum_{j=1}^{k-1}T(r_n,A_j)\right)+O(r_n^{\lambda-\omega_{j_0}+\varepsilon''})\right)=$$

$$O\left(r_n^{\omega_{j_0}}\sum_{j=1}^{k-1}T(r_n,A_j)+r_n^{\lambda+\varepsilon''}\right).$$

再联立(8)式可得

$$\frac{K}{q}\cdot\frac{T(r_n,A_0)}{\log r_n}<O(r_n^{\omega_{j_0}}\sum_{j=1}^{k-1}T(r_n,A_j)+r_n^{\lambda+\varepsilon''}).\quad(13)$$

因

$$\omega_{j_0}=\frac{\pi}{\beta_{j_0}-\alpha_{j_0}-2\varepsilon'}\leqslant\frac{\pi}{\beta_{j_0}-\alpha_{j_0}-\frac{4}{3}\varepsilon}\leqslant\omega_\varepsilon<\sigma,$$

$$\rho(A_j)=0,\lambda+\varepsilon''<\sigma.$$

由(6)式及(13)式可得

$$\sigma+2\varepsilon<\limsup_{n\to\infty}\log T(r_n,A_0)/\log r_n\leqslant$$
$$\max\{\omega_{j_0},\lambda+\varepsilon''\}<\sigma$$

矛盾.

(ii) 若 $\sigma=\rho$ 时,对 $A_0(z)$ 应用引理2,存在 σ 级 Pólya 峰序列 $\{r_n\}$,使

$$\liminf_{n\to\infty}\log T(r_n,A_0)/\log r_n\geqslant\sigma,$$

其中 $r_n\notin F$.

在(4)式和(7)式处用 σ 代替 $\sigma+2\varepsilon$,其余过程同(i)得 $\sigma<\sigma$ 矛盾.故综合(i)、(ii),定理1得证.

关于方程的无穷级解的研究,还有许多有意义的结果.如文[13-14]等就是对方程的某些系数给定一定的条件,证明了方程的非零解为无穷级.

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The Growth of Solutions of a Class Higher Order Linear Differential Equations in Angular Domains

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Abstract: The growth of solutions of the higher order differential equation $f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_0f=0$ is investigated in angular domains,where A_0 and $A_j(1\leqslant j\leqslant k-1)$ are meromorphic functions,assuming that A_0 has a finite deficient value a and $\rho(A_j)=0(1\leqslant j\leqslant k-1)$. When some conditions is given,it is proved that every solution $f\not\equiv0$ of the equation is of infinite order in some given angular domains.

Key words: differential equation; meromorphic function; deficient value; order in angular domains

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