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The Toeplitz Operators on the Weighted Banach Space of the Unit Ball

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Abstract: In the case of positive symbols, the continuity and compactness of a Toeplitz operator are characterized. The Toeplitz operator under investigation acts upon the weighted Banach space H_U^∞ which consists of analytic functions on the unit ball of C^n. Our characterizations are in terms of the Berezin transform.

Key words: Toeplitz operators; weighted Banach spaces; continuity; compactness

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0 Introduction and Results

For any integer $n \geqslant 1$, let C^n denote the Cartesian product of n copies of C. For $z = (z_1, \cdots, z_n)$ and $\zeta = (\zeta_1, \cdots, \zeta_n)$ in C^n, the inner product is defined by $\langle z, \zeta \rangle = z_1 \overline{\zeta_1} + z_2 \overline{\zeta_2} + \cdots + z_n \overline{\zeta_n}$, and throughout this paper, we denote $|z| = (z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n)^{1/2}$. Moreover, B^n stands for the open unit ball which consists of all z in C^n with $|z| < 1$.

Let $\mathrm{d}v$ denote the normalized volume measure on B^n (i.e., $v(B^n) = 1$). It is well known that, for a real parameter α,

$$\int_{B^n} (1 - |z|^2)^\alpha \mathrm{d}v(z) < \infty$$

holds if and only if $\alpha > -1$. We denote

$$\mathrm{d}v_\alpha(z) = a_\alpha (1 - |z|^2)^\alpha \mathrm{d}v(z)$$

where a_α is some positive constant satisfying $v_\alpha(B^n) = 1$ with some fixed $\alpha > -1$.

By L^p we denote the space of p integrable functions on B^n with respect to the measure $\mathrm{d}v_\alpha$. Here $1 \leqslant p \leqslant \infty$. The Bergman space L_a^p is the closed subspace of L^p which consists of all analytic functions. The normalized reproducing kernels for L_a^2 are of the form

$$k_z(\zeta) = \frac{(1 - |z|^2)^{(n+1+\alpha)/2}}{(1 - \langle z, \zeta \rangle)^{(n+1+\alpha)}}, |z| < 1, |\zeta| < 1.$$

For all $f \in L_a^2$, we have $\| k_z \| = 1$ and $\langle f, k_z \rangle = (1 - |z|^2)^{(n+1+\alpha)/2} f(z)$.

The orthogonal projection $P: L^2 \to L_a^2$ is defined by the following integral operator

$$Pf(z) = \int_{B^n} \frac{f(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+1+\alpha}} \mathrm{d}v_\alpha(\zeta), f \in L_a^2.$$

The Toeplitz operator on L_a^2 with symbol $\varphi \in L^1$ is defined by

$$T_\varphi f(z) = \int_{B^n} \frac{\varphi(\zeta) f(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+1+\alpha}} \mathrm{d}v_\alpha(\zeta).$$

The Berezin transform of a function $\varphi \in L^1$ is defined by

$$\tilde{\varphi}(z) = \int_{B^n} \frac{\varphi(\zeta)(1 - |z|^2)^{(n+1+\alpha)}}{|1 - \langle z, \zeta \rangle|^{2(n+1+\alpha)}} \mathrm{d}v_\alpha(\zeta). \quad (1)$$

Since the Bergman projection can be extended to L^1, the operator T_φ is well defined on $H^\infty(B^n)$, the space of bounded analytic functions on B^n, which is dense in L_a^2. Hence, T_φ is always densely defined on L_a^2.

For the case of bounded symmetric domains, the boundedness and compactness of Toeplitz operators on Bergman spaces were characterized in terms of Berezin transform in [1]. The Berezin transform was employed to make a study of Toeplitz operators on

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Bergman spaces of the unit ball in [2].

Recently, there arose an interest in studying operators on weighted Banach spaces of analytic functions. Motivated by [1-2,3-8] in this paper, in terms of Berezin transform, we will characterize the continuity and compactness of the Toeplitz operators on the weighted Banach space H_U of analytic functions on the unit ball. Denote

$$w(z) := 1 + |\log(1 - |z|)|, z \in B^n. \quad (2)$$

Then the space H_U^∞ (respectively, L_U^∞) consists of analytic (respectively, measurable) functions $f: B^n \to C$ such that for some nonnegative m and constant n

$$|f(z)| \leqslant A_m (w(z))^m, \text{for (almost) all } z \in B^n. \quad (3)$$

From [9-11] we know that the space H_U^∞ is not only a (LB)-space, i. e., countable inductive limit of Banach spaces, but also a complete space. More precisely, the topology could be defined by means of the family of weighted sup-seminorms

$$\|f\|_u = \sup_{z \in B^n} |f(z)| |u(z)|, u \in U,$$

where U is the set of all continuous, positive, radial functions $u: B^n \to \mathbf{R}$ such that for all m

$$|u(z)| \leqslant A_m (w(z))^{-m}.$$

The continuity and compactness of Toeplitz operators in H_U^∞ are characterized in terms of the growth properties of the Berezin transform as follows.

Theorem 1 Let φ be a nonnegative function defined on B^n. Then the Toeplitz operator $T_\varphi: H_U^\infty \to H_U^\infty$ is continuious if and only if there some k_0 and $A > 0$ such that the Berezin transform $\tilde{\varphi}$ in (1) satisfies

$$\tilde{\varphi}(z) \leqslant A (w(z))^{k_0}, z \in \mathrm{B}^n, \quad (4)$$

where $w(z)$ is defined in (2).

Theorem 2 Let φ be a nonnegative function defined on B^n. Then the Toeplitz operator $T_\varphi: H_U^\infty \to H_U^\infty$ is compact if there exist some k_0 such that for every positive m, there exists $A_k > 0$ with

$$\int_{B^n} \frac{\varphi(\zeta) (1 - |z|^2)^{(n+1+\alpha)} (u(\langle z,\zeta \rangle))^m}{|1 - \langle z,\zeta \rangle|^{2(n+1+\alpha)}} \mathrm{d}v_\alpha(\zeta) \leqslant A_k (w(z))^{k_0}, z \in B^n,$$

where $w(z)$ is defined in (2).

1 Preliminaries

For $z \in B^n$, let ψ_z be the analytic map of B^n onto B^n such that $\psi_z(0) = z$ and $\psi_z \circ \psi_z(w) = w$. These maps ψ_z are called involutions of B^n. For example, in the case of the unit disk,

$$\psi_z(w) = (z - w) / (1 - \bar{z}w)$$

is such a map.

The Bergman metric on the unit ball is given by

$$\beta(z,w) = \frac{1}{2} \log \frac{1 + |\psi_z(w)|}{1 - |\psi_z(w)|}.$$

For any $z \in B^n$ and $r > 0$ denote the Bergman metric ball by

$$D(z,w) = \{w \in B^n : \beta(z,w) < r\}.$$

And it is well-known that once r is fixed, then the volume $v_\alpha(D(z,r))$ is comparable to $(1 - |z|^2)^{n+1+\alpha}$. See [11] for example.

An r-lattice in the Bergman metric is a sequence $\{a_k\}$ in B^n satisfying the following conditions (see [2] for example)

(i) The unit ball is covered by the Bergman metric balls $\{D(a_k, r)\}$;

(ii) $\beta(a_i, a_j) \geqslant r/2$ for all i and j with $i \neq j$.

Throughout this paper, we will denote positive constants by A and it may be different at each occurrence.

We shall use the following lemma (see [11] and [2] for example)..

Lemma 1 Suppose that b is an arbitrary real number and $r > 0$. Then there is a positive constant A such that

$$\left| \frac{(1 - \langle z,\xi \rangle)^b}{(1 - \langle z,\eta \rangle)^b} \right| \leqslant A\beta(\xi,\eta)$$

for all z, ξ and η in B^n with $\beta(\xi,\eta) \leqslant r$.

As a consequence, if $r > 0$ and A is some positive constant, then the following inequality

$$A^{-1} \leqslant \left| \frac{1 - \langle z,\xi \rangle}{1 - \langle z,\eta \rangle} \right| \leqslant A \quad (5)$$

holds for all z, ξ and η in B^n with $\beta(\xi,\eta) \leqslant r$.

Remark 1 Let F and G be positive real valued functions. The symbol $F \cong G$ will be used, if there exist two absolute, positive constants A_1 and A_2 such that $A_1 F \leqslant G \leqslant A_2 F$ holds on the whole domain of definition. From (5), it is clear that

$$|1 - \langle z,\xi \rangle| \cong |1 - \langle z,\eta \rangle|$$

wherever z, ξ and η in B^n with $\beta(\xi,\eta) \leqslant r$.

We shall present some results on the space H_U^∞ for later use. Let $w(z)$ be defined in (2). And we define

$$U_m = \{f: f \in H_U^\infty \text{ and satisfies(3)}\},$$

the subsets U_m^P of L_U^∞ are defined in the same way. We know U that the sets U_m are bounded and even precompact in H_U^∞. Every bounded subsets of H_U^∞ is contained in a multiple of some U_m.

Lemma 2 (i) If $u \in U$, then the pointwise product $w^k v$ also belongs to U;

(ii) The mapping P is a continuious projection from L_U^∞ to L_U^∞;

(iii) For the projection P, $U_m^P \subset A_m U_{m+1}$ hold for all m.

We also collect some results on linear operators from [8-9] in the following lemma for later use.

Lemma 3 (i) A linear operator between two (LB)-spaces is continuous if and only if it maps bounded sets into bounded sets. In the case of this paper this means that $T_\varphi: H_U^\infty \to H_U^\infty$ is continuous if and only if for every $m \in \mathbf{N}$ one can find $A_m > 0$ and some exist some k_0 such that $T_\varphi(U_m) \subset A_m U_{m+k_0}$;

(ii) The linear operator $T_\varphi: H_U^\infty \to H_U^\infty$ is compact if and only if there exists some k_0 such that for every m, one can find $A_m > 0$ and such that $T_\varphi(U_m) \subset A_m U_{k_0}$.

2 Proof of Theorem

Proof of Theorem 1 Our proof follows from a combination of the methods and constructions in [8] and [2]. Let $\{a_k\}$ be a r-lattice which satisfies $|a_k| = 1 - 2^{-k}$ and $1 - 2^{-(k-1)} < r \leqslant 1 - 2^{-k}$. since $\varphi(\zeta)$ is a positive symbol, combitination of the definition of Berezin transform in (1) and (4) yields.

By Lemma 1, we have

$$\frac{(1 - |a_k|^2)^{(n+1+\alpha)}}{|1 - \langle a_k, \zeta\rangle|^{2(n+1+\alpha)}} \geqslant \frac{A}{v_\alpha(D(a_k, r))}.$$

Since $v_\alpha(D(a_k, r))$ is comparable to $(1 - |a_k|^2)^{n+1+\alpha}$, we have

$$\int_{D(a_k, r)} \varphi(\zeta)\, \mathrm{d}v_\alpha(\zeta) \leqslant \frac{Ak^{k_0}}{2^{k(n+1+\alpha)}}. \quad (6)$$

From Lemma 3, it is clear that the continuity of T_φ follows if for an arbitrary $m \in \mathbf{N}$ we can find a constant $A_{m,k_0} > 0$ such that $T_\varphi(U_m) \subset A_{m,k_0} U_{m+k_0+1}$.

To prove these facts, we fix the r-lattice $\{a_k\}$ in the Bergman metric as the beginning of the proof and estimate T_φ as follows. Without loss of generality, we may assume $z_N = 1 - 2^{-N}$, then

$$|T_\varphi f(z_N)| \leqslant \int_{B^n} \frac{\varphi(\zeta)\, |f(\zeta)|}{|1 - \langle z, \zeta\rangle|^{n+\alpha+1}} \mathrm{d}v_\alpha(\zeta). \quad (7)$$

According to lemma 1, $|1 - \langle z, \zeta\rangle|^{n+\alpha+1} \geqslant A\,|1 - \langle z_N, \zeta\rangle|^{n+\alpha+1}$ for $\beta(z, z_N) < r$, thus (7) can be written as

$$|T_\varphi f(z_N)| \leqslant \sum_{k=1}^{\infty} A \int_{D(a_k, r)} \frac{\varphi(\zeta)\, |f(\zeta)|}{|1 - \langle z_N, \zeta\rangle|^{n+\alpha+1}} \mathrm{d}v_\alpha(\zeta). \quad (8)$$

For $\zeta \in D(a_k, r)$, we have $|1 - \langle z_N, \zeta\rangle| \cong 1 - (1 - 2^{-N})(1 - 2^{-k}) \cong 2^{-N} + 2^{-k}$. Since $f \in U_m$, we have the following estimate

$$|f(\zeta)| \leqslant A_m k^m. \quad (9)$$

By (8) and (9), (7) can be bounded by a constant times

$$\sum_{k=1}^{\infty} \frac{k^m}{(2^{-N} + 2^{-k})^{n+1+\alpha}} \int_{D(a_k, r)} \varphi(\zeta)\, \mathrm{d}v_\alpha(\zeta).$$

Combination of these estimates and (6) yield

$$|T_\varphi f(z_N)| \leqslant A \sum_{k=1}^{\infty} \frac{k^{(k_0+m)}}{2^{k(n+1+\alpha)}(2^{-N} + 2^{-k})^{n+1+\alpha}}. \quad (10)$$

Now we proceed with the estimate of the series

$$\sum_{k=1}^{\infty} \frac{k^{k_0+m}}{2^{k(n+1+\alpha)}(2^{-N} + 2^{-k})^{n+1+\alpha}} \leqslant \sum_{k=1}^{\infty} \frac{k^{(k_0+m)}}{1 + 2^{k-N}}. \quad (11)$$

We write the series on the right side of (11) as

$$\sum_{k=1}^{\infty} \frac{k^{(k_0+m)}}{1 + 2^{k-N}} = \sum_{k \leqslant N} \frac{k^{(k_0+m)}}{1 + 2^{k-N}} + \sum_{k > N} \frac{k^{(k_0+m)}}{1 + 2^{k-N}}. \quad (12)$$

It is easy to derive the following inequality

$$\sum_{k \leqslant N} \frac{k^{(k_0+m)}}{1 + 2^{k-N}} \leqslant \sum_{k \leqslant N} k^{(k_0+m)} \leqslant AN^{k_0+m+1}. \quad (13)$$

Integrating by parts the expression $\int_N^\infty x^{k_0-m} \mathrm{e}^{-x} \mathrm{d}x$ yields

$$\sum_{k \geqslant N} \frac{k^{(k_0+m)}}{1 + 2^{k-N}} \leqslant A(k_0 + m + 1)!\, N^{k_0+m+1}. \quad (14)$$

By (10) ~ (14), we get the estimate

$$|T_\varphi f(z_N)| \leqslant AN^{k_0+m+1} \leqslant A'\, |\log(1 - |z_N|)|^{k_0+m+1},$$

which shows that T_φ maps U_m into U_{k_0+m+1}. Thus the continuity of T_φ is proved.

Conversely, if T_φ is bounded. we can find $k_0 \in \mathbf{N}$ such that T_φ maps U_1 into AU_{k_0}. For every $z \in B^n$, we have

$$\| (1 - |z|^2)^{n+1+\alpha} K_z(\omega) \|_\infty \leqslant 2^{n+1}.$$

Thus, $T_\varphi((1 - |z|^2)^{n+1+\alpha} K_z(\omega)) \in 2^{n+1+\alpha} AU_{k_0}$ for every z and

$$|T_\varphi((1-|z|^2)^{n+1+\alpha}K_z(\omega))| \leqslant 2^{n+1+\alpha}Aw(\omega)^{k_0}$$

holds for all $\zeta \in B^n$. Taking $\omega = z$, we get

$$|\tilde{\varphi}(z)| = \left|\int_{B^n} \frac{\varphi(\zeta)(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,\zeta\rangle|^{2(n+1+\alpha)}} dv_\alpha(\zeta)\right| =$$

$$\left|\int_{B^n} \frac{\varphi(\zeta)}{(1-\langle z,\zeta\rangle)^{(n+1+\alpha)}} \frac{(1-|z|^2)^{n+1+\alpha}}{(1-\langle z,\zeta\rangle)^{(n+1+\alpha)}} dv_\alpha(\zeta)\right| =$$

$$|T_\varphi((1-|z|^2)^{n+1+\alpha}K_z)(\omega)| \leqslant Aw(z)^{k_0}.$$

Proof of Theorem 2 Since the proof is the same to the proof of Theorem 1, we just need to give a sketch of the proof here.

Let $\{a_k\}$ be a r-lattice which satisfies $|a_k| = 1-2^{-k}$ and $1-2^{k-1} < r \leqslant 1-2^{-k}$. Applying the same reasoning of (6), with

$$|\log(1-|\langle z,\zeta\rangle|)| \cong -\log(1-|\langle z,\zeta\rangle|) \cong k$$

included, we have

$$\int_{D(a_k,r)} \varphi(\zeta)\, dv_\alpha(\zeta) \leqslant \frac{Ak^{k_0-m}}{2^{2(n+1+\alpha)}}. \qquad (15).$$

From Lemma 3, it is clear that the compactness of T_φ follows if we can show that there exists $m_0 \in \mathbf{N}$ such that for every $m \in \mathbf{N}$ one can find $A_m > 0$ and such that $T_\varphi(U_m) \subset A_m U_{m_0}$.

To prove these facts, we fix the r-lattice $\{a_k\}$ in the Bergman metric as the beginning of the proof and estimate T_φ as follows. Without loss of generalization, we may assume $z_N = 1-2^{-N}$, replacing (6) by (15), then applying the reasoning of (7) ~ (14), we have the estimate

$$|T_\varphi f(z_N)| \leqslant AN^{k_0+1} \leqslant A'|\log(1-|z_N|)|^{k_0+1},$$

which shows that T_φ maps U_m into U_{k_0+1}. Thus the compactness of T_φ is proved.

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单位球上加权 Banach 空间中的 Toeplitz 算子

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摘要: 应用 Berezin 变换的方法对单位球上加权 Banach 空间中的 Toeplitz 算子进行刻画,对正函数的情形,将现有的单位圆的相关结论推广至单位球,得到 Toeplitz 算子连续的充分必要条件,并给出 Toeplitz 算子为紧算子的充分条件.

关键词: Toeplitz 算子; 加权 Banach 空间; 连续性; 紧致性

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