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The New Developments in the Research of Nonlinear Complex Differential Equations

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Abstract: The problem that an algebraic differential equation has no admissible meromorphic solution is studied by using Nevanlinna's theory and Wiman-Valiron theory. The structures of the entire solutions of some nonlinear differential equations are given, and the Hayman's theorems to some differential polynomials are extended by using these results. Finally, a survey of his groups' recent researches about non-linear complex differential equations and their applications is given.

Key words: Nevanlinna's value distribution theory; nonlinear differential equation; differential polynomial; meromorphic solution; entire solution

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0 Nevanlinna Theory

Nevanlinna theory is a useful tool in the studying of the complex differential equations. In this section, we introduce the basic concepts and notations in Nevanlinna theory. For further results, we suggest the author looking for [1-5]. In this paper, we always assume that f is a nonconstant meromorphic function in the complex plane C. We use $n(r,1/(f-a))$ to denote the numbers of roots of $f(z)=a$ on $|z|\leqslant r$ counting multiplicity, $n(r,f)$ to denote the numbers of poles of $f(z)$ on $|z|\leqslant r$ counting multiplicity. The counting functions of f are defined as following:

$$N(r,1/(f-a)) = \int_0^r \frac{n(t,1/(f-a))-n(0,1/(f-a))}{t}\mathrm{d}t + n(0,1/(f-a))\log r,$$

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}\mathrm{d}t + n(0,f)\log r.$$

Furthermore, denote $m(r,f)$ and $m(r,1/(f-a))$ $(a\neq\infty)$ as

$$m(r,1/(f-a)) = \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f(re^{\mathrm{i}\theta})-a}\right|\mathrm{d}\theta,$$

$$m(r,f) = \int_0^{2\pi}\log^+|f(re^{\mathrm{i}\theta})|\mathrm{d}\theta,$$

where $\log^+ x=\max\{\log x,0\}$. Nevanlinna characteristic function of a meromorphic function f will be defined as $T(r,f)=m(r,f)+N(r,f)$. $T(r,f)$ is a non-negative increasing function for nonconstant meromorphic function f. If f is transcendental, then $T(r,f)/\log r\to\infty$ as $r\to\infty$. If f is a rational function, then it is easy to show that $T(r,f)=\deg f\log r+O(1)$.

By applying Jesen formula, it is easy to deduce the First Fundamental Theorem.

Theorem 1 (First Fundamental Theorem) Let $f(z)$ be meromorphic in $|z|<R(\leqslant\infty)$. If a is an arbitrary complex number and $0<r<R$, and

$$f(z)-a=\sum_{i=m}^{\infty}c_i z^i, c_m\neq 0, m\in\mathbf{Z}$$

is the Laurent expansion of $f-a$ at the origin, then we have

$$T(r,f) = T(r,1/(f-a)) + \log|c_m| + \varepsilon(r,a),$$

where $|\varepsilon(r,a)|\leqslant\log 2+\log^+ a$.

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Now we state Nevanlinna's Second Fundamental Theorem, which is the most important in Nevanlinna theory.

Theorem 2 (Second Fundamental Theorem) Let f be a non-constant meromorphic function, let $q \geqslant 2$ and let $a_1, a_2, \cdots, a_q \in \mathbf{C}$ be distinct points. Then

$$m(r,f) + \sum_{i=1}^{q} m(r, 1/(f-a_i)) \leqslant$$
$$2T(r,f) - N_1(r) + S(r,f),$$

where

$$N_1(r) = 2N(r,f) - N(r,f') + N(r,1/f'),$$

and

$$S(r,f) = O(\log [rT(r,f)]) \ (r \to \infty),$$

except for a set E with a finite linear measure.

Definition 1 The order of a meromorphic function f is defined by

$$\rho = \rho(f) = \overline{\lim_{r\to\infty}} \log T(r,f) / \log r.$$

Definition 2 The lower order of a meromorphic function f is defined by

$$\lambda = \lambda(f) = \underline{\lim_{r\to\infty}} \log T(r,f) / \log r.$$

Definition 3 We say that a meromorphic function $a(z)$ is a small function of $f(z)$ if

$$T(r,a) = o\{T(r,f)\}, r \to \infty, r \notin E,$$

where E is a set of r with a finite linear measure.

The following logarithmic derivative Lemma is very useful in the study of complex differential equations.

Theorem 3 [4] Let $f(z)$ be a nonconstant meromorphic function, Then $m(r, f'/f) = O(\log r)$ $(r \to \infty)$, if f is of finite order, and $m(r, f'/f) = O(\log(rT(r,f)))$ $(r \to \infty)$, possibly outside a set E of r with finite linear measure if $f(z)$ is of infinite order.

1 Wiman-Valiron Theory

Wiman-Valiron theory is useful in the study of entire functions. Let f be an entire function and its Taylor expansion be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We define the central index and the maximum term of an entire function f, $\upsilon(r,f)$, $\mu(r,f)$ as following:

$$\mu(r,f) = \max_n |a_n| r^n,$$

$$\upsilon(r,f) = \max\{n, |a_n| r^n = \mu(r,f)\}.$$

If $r = 0$, then $\upsilon(r,f) = p$, where a_p is the first nonzero coefficient in the Taylor expansion of f.

Example 1 $f(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, then $\mu(r, f) = r^{[r]} / [r]!$ and $\upsilon(r,f) = [r]$, where $[r]$ is the largest integer not greater than r.

Theorem 4 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a nonconstant entire function. Then, we have the following three conclusions:

(i) There exists a non-negative real number r_0 such that $\mu(r,f)$ is a strictly increasing function of r for $r \geqslant r_0$, and $\mu(r,f) \to \infty$ as $r \to \infty$;

(ii) $\upsilon(r,f)$ is a nondecreasing function of r, furthermore, if $f(z)$ is transcendental, then $\upsilon(r,f) \to \infty$ as $r \to \infty$ and $\upsilon(r,f)$ is right continuous;

(iii) $\mu(r,f)$ is a continuous function of r.

Theorem 5 (Wiman-Valiron Theorem [3]) Let f be a transcendental entire function, and $0 < \delta < 1/4$. Suppose that at the point z with $|z| = r$ the inequality

$$|f(z)| > M(r,f)\, \upsilon(r,f)^{-1/4+\delta}$$

holds. Then there exists a set F in $\mathbf{R}^+$ of finite logarithmic measure, i.e., $\int_F 1/t \, dt < +\infty$ such that

$$f^{(m)}(z) = (\upsilon(r,f)/z)^m (1 + o(1)) f(z)$$

holds whenever m is a fixed nonnegative integer and $r \notin F$.

2 Admissible Meromorphic Solutions of Algebraic Differential Equations

In general, a nonlinear algebraic differential equation is of the form:

$$P(z, f, f', \cdots, f^{(n)}) = 0, \quad (1)$$

where P is a polynomial in f and its derivatives with meromorphic coefficients. One can rewrite equation (1) in the form:

$$\sum_{\lambda \in I} a_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n} = 0, \quad (2)$$

where I is a finite set of multi-indices $(\lambda_0, \lambda_1, \cdots, \lambda_n) = \lambda$ and $a_\lambda(z)$ is a meromorphic function. We define a differential monomial in f as

$$M_\lambda[z,f] = a_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(n)})^{\lambda_n}.$$

The degree γ_{M_λ} and the weight Γ_{M_λ} of M_λ are defined by

$$\gamma_{M_\lambda} = \lambda_0 + \lambda_1 + \cdots + \lambda_n,$$

$$\Gamma_{M_\lambda} = \lambda_0 + 2\lambda_1 + \cdots + (n+1)\lambda_n.$$

Thus the left hand side of (2) can be expressed as a finite sum of differential monomials and which will be called a differential polynomial in f, i. e.,

$$P[z,f] = P(z,f,f',\cdots,f^{(n)}) = \sum_{\lambda\in I} M_\lambda[z,f].$$

The degree γ_P and the weight Γ_P of P are defined by $\gamma_P = \max_{\lambda\in I}\gamma_{M_\lambda}$, $\Gamma_P = \max_{\lambda\in I}\Gamma_{M_\lambda}$.

We say that the term $M_\lambda[z,f]$ is a dominant term of $P[z,f]$ if $\gamma_{M_\lambda} = \gamma_P$. Obviously, a differential polynomial may have more than one dominant term. A meromorphic solution f of equation (2) is called admissible, if $T(r,\alpha_\lambda) = S(r,f)$ holds for all coefficients $\alpha_\lambda(z)$, $\lambda\in I$.

In 1980, Gackstatter and Laine [6] considered the special algebraic differential equation of the following form, $(f')^n = Q(z,f)$, where $Q(z,f)$ is a polynomial in f with meromorphic coefficients, and conjectured that it does not possess any admissible solution when

$$q = \deg_f Q(z,f) \leqslant n-1.$$

This conjecture attracts many researchers' interest [7-12]. In 1990, He and Laine [7] gave a positive answer to the above conjecture. One year later, Ishizaki [8] proved the following more general result.

Theorem 6 The differential equation $P(z,f') = Q(z,f)$, where $P(z,f')$, resp. $Q(z,f)$, is a polynomial in f, with meromorphic coefficients such that $1\leqslant q := \deg_f Q(z,f) \leqslant p-1 := \deg_{f'} P(z,f') - 1$ admits no admissible solutions.

For general algebraic differential equation, Wittich [13] gave a classic result as follows.

Theorem 7 If the algebraic differential equation $P(z,f) = 0$, where $P(z,f)$ is a differential polynomial in f with polynomial coefficients, has only one dominant term, then the equation has no transcendental entire solutions.

The following theorem is an extending result of Theorem 6 and Theorem 7.

Theorem 8 [14] If the algebraic differential equation

$$P(z,f) = 0, \qquad (3)$$

where $P(z,f)$ is a differential polynomial in f with meromorphic coefficients, has only one dominant term, then equation (3) has no admissible transcendental meromorphic solutions satisfying

$$N(r,f) = S(r,f).$$

The following two examples show that conditions $P(z,f)$ has only one dominant term and $N(r,f) = S(r,f)$ in Theorem cannot be dropped.

Example 2 The following differential equation

$$(f')^2 + 2ff'/z + (1+1/z^2)f^2 - 1/z^2 = 0 \qquad (4)$$

has an admissible transcendental meromorphic solution $f(z) = (\cos z)/z$ satisfying $N(r,f) = S(r,f)$. However, equation (4) has three dominant terms.

Example 3 The meromorphic function satisfies $f(z) = \tan(z^2)$ the following algebraic differential equation

$$(f')^2 f'' - 4z(f')^3 f - (f')^3/z + f'' = 8z^2 f^3 + 2f^2 + 8z^2 f + 2. \qquad (5)$$

Equation (5) has only one dominant term, but the counting function

$$N(r,f) = T(r,f) + S(r,f).$$

The following result is the extension of Theorem 6.

Corollary 1 [14] The differential equation

$$P(z,f^{(k)}) = Q(z,f),$$

where $P(z,f^{(k)})$, resp. $Q(z,f)$, is a polynomial in $f^{(k)}$, resp. in f, with meromorphic coefficients such that $q := \deg_f Q(z,f) \leqslant p-1 := \deg_{f^{(k)}} P(z,f^{(k)}) - 1$ and $k\geqslant 1$ is a positive integer, has no admissible transcendental meromorphic solutions.

3 A Certain Type of Nonlinear Differential Equations

Some mathematicians studied the non-linear differential equations of the form:

$$f^n + P_d(z,f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $P_d(z,f)$ denotes a polynomial in f and its derivatives with a total degree $d\leqslant n-1$, with small functions of f as the coefficients, $p_1(z)$, $p_2(z)$ are two nonzero polynomials and α_1, α_2 are two nonzero constants [15-18]. Moreover, $P_d(z,f)$ is called an algebraic differential polynomial in f, if all its coefficients are polynomials in z.

Recently, it is shown in [18] that the equation

$$4f^3(z) + 3f''(z) = -\sin 3z$$

has exactly three nonconstant entire solutions, namely

$$f_1(z) = \sin z, f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z,$$

$$f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z.$$

More recently, the following two results have been obtained:

Theorem 9[15] Let $k \geqslant 1$ be an integer and $P_d(z,f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leqslant n-3$ with small functions of f as the coefficients. If $p_1(z)$, $p_2(z)$ are two nonzero polynomials and α_1,α_2 are two nonzero constants such that α_1/α_2 is not rational, then the equation

$$f^n + P_d(z,f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}$$

does not have any transcendental entire solution.

Theorem 10[19] Let $n \geqslant 2$ be an integer, $P_d(z,f)$ be an algebraic differential polynomial in $f(z)$ of degree $d \leqslant n-2$ with small functions of f as the coefficients, and $p_1,p_2,\alpha_1,\alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$. If f is a transcendental meromorphic solution of the following equation

$$f^n(z) + P_d(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

and satisfying $N(r,f) = S(r,f)$, then one of the following holds:

(i) $f(z) = c_0 + c_1 e^{\alpha_1 z/n}$;

(ii) $f(z) = c_0 + c_2 e^{\alpha_2 z/n}$;

(iii) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$, and $\alpha_1 + \alpha_2 = 0$,

where c_0 is a small function of $f(z)$ and c_1,c_2 are constants satisfying $c_1^n = p_1, c_2^n = p_2$.

In [20], Liao, Yang and Zhang obtained the following results.

Theorem 11 Let $n \geqslant 3$ and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that p_1,p_2 are rational functions and α_1,α_2 are polynomials. If $d \leqslant n-2$, the following differential equation

$$f^n + Q_d(z,f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z} \qquad (6)$$

admits a meromorphic function f with finitely many poles. Then α'_1/α'_2 is a rational number.

Furthermore, only one of the following four cases holds:

(i) $f(z) = q(z)e^{P(z)}$ and $\alpha'_1/\alpha'_2 = 1$, where $q(z)$ is a rational function and $P(z)$ is a polynomial with $nP'(z) = \alpha'_1 = \alpha'_2$;

(ii) $f(z) = q(z)e^{P(z)}$ and either $\alpha'_1/\alpha'_2 = k/n$ or $\alpha'_1/\alpha'_2 = n/k$, where $q(z)$ is a rational function, k is an integer with $1 \leqslant k \leqslant n$ and $P(z)$ is a polynomial with $nP'(z) = \alpha'_1$ or $nP'(z) = \alpha'_2$;

(iii) f satisfies the first order linear differential equation

$$f' = (p'_2/(np_2) + \alpha'_2/n)f + \psi$$

and $\alpha'_1/\alpha'_2 = (n-1)/n$ or f satisfies the first order linear differential equation

$$f' = (p'_1/(np_1) + \alpha'_1/n)f + \psi$$

and

$$\alpha'_1/\alpha'_2 = n/(n-1),$$

where ψ is a rational function;

(iv) $f(z) = \gamma_1(z)e^{\beta_1(z)} + \gamma_2(z)e^{-\beta_1(z)}$ and $\alpha'_1/\alpha'_2 = -1$, where γ_1,γ_2 are rational functions and $\beta_1(z)$ is a polynomial with $n\beta'_1 = \alpha'_1$ or $n\beta'_1 = \alpha'_2$.

Remark 1 The four cases in the theorem exist. For instance, $f = e^z + z + 1$ solves the following non-linear differential equation

$$f^3 - 2(z+1)^2 f'' - (z+1)^2 f = e^{3z} + 3(z+1)e^{2z}.$$

This example shows the case (3) in the theorem certainly exists.

Corollary 2 Let $n \geqslant 3$ and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that p_1,p_2 are rational functions and α_1,α_2 are constants. If $d \leqslant n-2$, the following differential equation

$$f^n + Q_d(z,f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}$$

admits a meromorphic function f with finitely many poles. Then α_1/α_2 is a rational number.

Furthermore, only one of the following four cases holds:

(i) $\alpha_1/\alpha_2 = 1$ and $f(z) = q(z)e^{\alpha_1 z/n}$, where

$$q(z)^n = p_1(z) + p_2(z)$$

is a rational function;

(ii) $\alpha_1/\alpha_2 = n/k$ for some $1 \leqslant k \leqslant d$ and $f(z) = q(z)e^{\alpha_1 z/n}$, where $q(z)^n = p_1(z)$ or $\alpha_1/\alpha_2 = k/n$ for some $1 \leqslant k \leqslant d$ and $f(z) = q(z)e^{\alpha_2 z/n}$, where $q(z)^n = p_2(z)$;

(iii) $\alpha_1/\alpha_2 = (n-1)/n$ and f satisfies the first order linear differential equation

$$f' = (p'_2/(np_2) + \alpha_2/n)f + \psi$$

or

$$\alpha_1/\alpha_2 = n/(n-1)$$

and f satisfies the first order linear differential equation

$$f' = (p'_1/(np_1) + \alpha_1/n)f + \psi,$$

where ψ is a rational function;

(iv) $\alpha_1 + \alpha_2 = 0$ and $f(z) = q_1(z)e^{\alpha_1 z/n} + q_2(z)e^{-\alpha_1 z/n}$ where $q_1(z)^n = p_1(z)$ and $q_2(z)^n = p_2(z)$.

Theorem 12 Let $n \geqslant 3$ and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that R, p_1, p_2 are rational functions and α_1, α_2 are polynomials. If $d \leqslant n-2$ and the following differential equation

$$f^n + R(z)f^{n-1} + Q_d(z,f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z} \quad (7)$$

admits a meromorphic function f with finitely many poles. Then α'_1/α'_2 is a rational number.

Furthermore, only one of the following four cases holds:

(i) $f(z) = -R(z)/n + q(z)e^{P(z)}$ and $\alpha'_1/\alpha'_2 = 1$, where $q(z)$ is a rational function with and $P(z)$ is a polynomial with $nP'(z) = \alpha'_1 = \alpha'_2$;

(ii) $f(z) = -R(z)/n + q(z)e^{P(z)}$ and either $\alpha'_1/\alpha'_2 = k/n$ or $\alpha'_1/\alpha'_2 = n/k$, where $q(z)$ is a rational function, k is an integer with $1 \leqslant k \leqslant n$ and $P(z)$ is a polynomial with $nP'(z) = \alpha'_1$ or $nP'(z) = \alpha'_2$;

(iii) f satisfies the first order linear differential equation

$$f' = (p'_2/(np_2) + \alpha'_2/n)f + \psi$$

and

$$\alpha'_1/\alpha'_2 = (n-1)/n$$

or f satisfies the first order linear differential equation

$$f' = (p'_1/(np_1) + \alpha'_1/n)f + \psi$$

and $\alpha'_1/\alpha'_2 = n/(n-1)$, where ψ is a rational function;

(iv) $f(z) = -R(z)/n + \gamma_1(z)e^{\beta_1(z)} + \gamma_2(z) \cdot e^{-\beta_1(z)}$ and $\alpha'_1/\alpha'_2 = -1$, where γ_1, γ_2 are rational functions and $\beta_1(z)$ is a polynomial with $n\beta'_1 = \alpha'_1$ or $n\beta'_1 = \alpha'_2$.

Corollary 3 Let $n \geqslant 3$ and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that R, p_1, p_2 are rational functions and α_1, α_2 are constants. If $d \leqslant n-2$, the following differential equation

$$f^n + R(z)f^{n-1} + Q_d(z,f) = p_1(z)e^{\alpha_1 z} + p_2(z)e^{\alpha_2 z}$$

admits a meromorphic function f with finitely many poles. Then α_1/α_2 is a rational number.

Furthermore, only one of the following four cases holds:

(i) $\alpha_1/\alpha_2 = 1$ and $f(z) = R(z)/n + q(z)e^{\alpha_1 z/n}$, where $q(z)^n = p_1(z) + p_2(z)$ is a rational function;

(ii) $\alpha_1/\alpha_2 = n/k$ for some $1 \leqslant k \leqslant d$ and $f(z) = R(z)/n + q(z)e^{\alpha_1 z/n}$, where $q(z)^n = p_1(z)$ or $\alpha_1/\alpha_2 = k/n$ for some $1 \leqslant k \leqslant d$ and $f(z) = R(z)/n + q(z)e^{\alpha_1 z/n}$, where $q(z)^n = p_2(z)$;

(iii) $\alpha_1/\alpha_2 = (n-1)/n$ and f satisfies the first order linear differential equation $f' = (p'_2/(np_2) + \alpha_2/n)f + \psi$ or $\alpha_1/\alpha_2 = n/(n-1)$ and f satisfies the first order linear differential equation

$$f' = (p'_1/(np_1) + \alpha_1/n)f + \psi,$$

where ψ is a rational function;

(iv) $\alpha_1 + \alpha_2 = 0$ and

$$f(z) = -R(z)/n + q_1(z)e^{\alpha_1 z/n} + q_2(z)e^{-\alpha_1 z/n},$$

where $q_1(z)^n = p_1(z)$ and $q_2(z)^n = p_2(z)$.

The Sketch of the Proof of Theorem 11

Step 1 Reduce the degree of the equation (6) by using the following Clunie lemma.

Lemma 1 (Clunie Lemma[21,3]) Let $f(z)$ be meromorphic and transcendental function in the plane and satisfy $f^n(z)P(f) = Q(f)$, where $P(f), Q(f)$ are differential polynomials in $f(z)$ with rational functions as the coefficients and the degree of $Q(f)$ is at most n, then $m(r, P(f)) = O(\log r)$ $(r \to \infty)$, if f is of finite order, and $m(r, P(f)) = O(\log(rT(r,f)))$ $(r \to \infty)$ possibly outside a set E of r with finite linear measure if $f(z)$ is of infinite order.

By differentiating the equation (6) and eliminating $e^{\alpha_1(z)}, e^{\alpha_2(z)}$, respectively, from the equations, we have

$$(h_1(z)f^2 + h_2(z)ff' + h_3(z)f'^2 + h_4(z)ff'') \cdot f^{n-2} = Q_d^*(z,f).$$

By Clunie Lemma, the differential equation is reduced into the following differential equation

$$h_1(z)f^2 + h_2(z)ff' + h_3(z)f'^2 + h_4(z)ff'' = a(z),$$

where $h_i(z)$ $(i = 1,2,3,4)$, $a(z)$ are rational functions.

Steps 2 Linearization

By using the following main lemma, we can reduce the above equation into a second order linear differential equation.

Lemma 2 Let q_1, q_2, q_3, a be rational functions and $q_3 a \neq 0$. If the differential equation

$$q_1(z)f^2 + q_2(z)ff' + q_3(z)f'^2 = a(z) \quad (8)$$

admits a transcendental meromorphic solution, then

(i) any meromorphic solution of (8) must be of finite order, and

(ii) the following identity holds:

$$q_3(q_2^2 - 4q_1q_3)\frac{a'}{a} + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q'_3 \equiv 0,$$

and any transcendental meromorphic solution f of the equation (8) satisfies the following linear second order differential equation

$$f'' = \left(\frac{a'}{2a} - \frac{q_3'}{2q_3} - \frac{q_2}{2q_3}\right) f' - \frac{1}{q_2}\left(q_1' - q_1 \frac{a'}{a}\right) f.$$

Furthermore, if $q_2^2 - 4q_1 q_3 \neq 0$ and $\deg_\infty q_2 \geqslant \deg_\infty q_3$, then the differential equation (8) has no transcendental meromorphic solution.

The Sketch of the Proof of Theorem 12

Assume that f is a meromorphic solution with only finitely many poles of the equation (7). Let $g(z) = f(z) + R(z)/n$. Then, g is a transcendental meromorphic function with only finitely many poles and satisfies the following differential equation

$$f^n + Q_{n-2}^*(z, f) = p_1(z) e^{\alpha_1(z)} + p_2(z) e^{\alpha_2(z)},$$

Where $Q_{n-2}^*(z, f)$ is a differential equation with degree $d \leqslant n-2$. The conclusions of the theorem follows immediately from Theorem 11.

4 Nonhomogeneous Algebraic Differential Equations and Hayman's Theorems of Differential Polynomials

It is always an essential problem to find out a structure of solutions to any differential equations. In 1980, F. Gackstatter and I. Laine [22] conjectured that the algebraic differential equation $(f')^n = p_m(f)$, where $p_m(f)$ is a polynomial in f and n is a positive integer, does not possess any admissible solution when $m \leqslant n-1$. In 1990, Y. He and I. Laine [7] gave a positive answer to the conjecture. Recently, J. Zhang and Liao [14] proved that if the algebraic differential equation with polynomial coefficients

$$Q_d(z, f) = 0 \qquad (9)$$

has only one dominant term (highest degree term), then (9) has no admissible transcendental meromorphic solutions with a few poles. There are also many other papers concerning the structure of solutions to various differential equations [19-20,23-27]. Recently, Liao-Ye [28] consider the algebraic differential

$$f^n f' + Q_d(z, f) = u(z) e^{v(z)}, \qquad (10)$$

where $Q_d(z, f)$ be a differential polynomial in f with $n \geqslant d+1$ and rational function coefficients, u is a nonzero rational function and v is a non-constant polynomial. Clearly, $f^n f'$ is the only dominant term in (10) and its nonhomogeneous term is a transcendental meromorphic function. Thus, Liao-Ye find a simple and neat expression for meromorphic solutions to (10) if the solutions have a few poles. This also means the solution has finitely many zeros and determined by the term ue^v in the differential equation. Further, the result can be used to generalize a theorem of Hayman in [29].

Hayman [29] proved that if f is a transcendental entire function, then $f'f^n$ assumes every non-zero complex number infinitely many times, provided that $n \geqslant 2$. Since then, there are many research publications [30-34] regarding this type of Picard-value problem. For example, Mues [32] extended the result proving that if $p(f)$ is a non-constant polynomial in f, then $p(f) f'$ assumes every non-zero complex number infinitely many times. Zhang and Li [34] proved that if f is a transcendental meromorphic function with $N(r, f) = S(r, f)$ and p a polynomial with degreed $d \geqslant 1$, then $p(f) f'$ takes every non-zero complex number infinitely many times. Recently, Liao-Ye [28] prove that if p_k, q_m are two polynomials with degree $k \geqslant m+1$ and f a transcendental entire function, then $p_k(f) f' + q_m(f)$ assumes every complex number, with possible one exception value, infinitely many times. More interesting, we show that if $p_k(f) \cdot f' + q_m(f)$ takes the exceptional value finitely many times, then we can prove that q_m has to be a constant polynomial and p_n is a complete power function, or, $f(z) = Ae^{Bz} + C$ where A, B, C are constant. Liao [35] consider differential equation

$$f^n + Q_d(z, f) = p(z) e^{\alpha(z)}, \qquad (11)$$

where $Q_d(z, f)$ be a differential polynomial in f of degree $d \leqslant n-2$ with rational functions as its coefficients, p is a nonzero rational function, α is a nonconstant polynomial. He got if the equation (11) has a meromorphic solution f with finitely many poles, then f must has finitely many zeros. Furthermore, this result can be used to generalize a theorem of Hayman in [29].

Theorem 13 [28] Let $Q_d(z, f)$ be a differential polynomial in f of degree d with rational function coefficients. Suppose that u is a non-zero rational function and v is a non-constant polynomial. If $n \geqslant d+1$ and the following differential equation

$$f^n f' + Q_d(z, f) = u(z) e^{v(z)}$$

admits a meromorphic function f with finitely many poles, then f has the following form

$$f(z) = s(z)e^{v(z)/(n+1)}, Q_d(z,f) \equiv 0,$$

where $s(z)$ is a rational function with

$$s^n((n+1)s' + v's) = (n+1)u.$$

In particular, if u is a polynomial, then s is a polynomial, too.

Remark 2　The condition $n \geqslant d+1$ in the theorem is necessary. For example, $f(z) = e^z + z$ solve the following differential equation

$$ff'^2 - 2zf'^2 - z^2 f' = e^{3z},$$

where $n = d = 2$.

Theorem 14 [28]　Let f be a transcendental entire function, $q_m(f) = b_m f^m + \cdots + b_1 f + b_0$ a polynomial with degree m and n a positive integer with $n \geqslant m+1$. Then $f'f^n + q_m(f)$ assumes every complex number α infinitely many times, except a possible value $b_0 = q_m(0)$. On the other hand, if $f'f^n + q_m(f)$ assumes $b_0 = q_m(0)$ finitely many times, then $q_m(z) \equiv b_0$, f and f' have only finitely many zeros.

Remark 3　The restrict condition $n \geqslant m+1$ in the theorem is necessary. For instance, if $f(z) = e^z + 1$ and $q_2(z) = -2z^2 + 3z$, then $f'f^2 - 2f^2 + 3f = e^{3z} + 1$ does not assume $1 \neq q_2(0)$.

Theorem 15 [28]　Let f be a transcendental entire function,

$$p_n(f) = a_n f^n + \cdots + a_1 f + a_0$$

a polynomial with degree n,

$$q_m(f) = b_m f^m + \cdots + b_1 f + b_0$$

is a polynomial with degree m and $n \geqslant m+1$. Then $f'p_n(f) + q_m(f)$ assumes every complex number α infinitely many times, except a possible value $q_m(-a_{n-1}/(na_n))$. On the other hand, if $f'p_n(f) + q_m(f)$ assumes the complex value $q_m(-a_{n-1}/(na_n))$ finitely many times, then either

(i) $p_n(z) = a_n(z + a_{n-1}/(na_n))^n$, $q_m(z)$ is a constant polynomial, which is $q_m(-a_{n-1}/(na_n))$; and $f + a_{n-1}/(na_n)$, f' have only finitely many zeros; or

(ii) $f(z) = Ae^{Bz} + a_{n-1}/(na_n)$, where A, B are some constants; only when q_m is non-constant and f is of finite order.

Remark 4　Theorem 14 is a special case of Theorem 15. But, we need Theorem 14 in the proof of Theorem 15.

Remark 5　It is challenge to prove that Theorem 14 and/or Theorem 15 are valid for meromorphic functions in the complex plane.

Example 4　If $f(z) = e^z$, then

$$f'(f^3 - f) + f^2 = e^{3z}$$

does not assume $0 = q_2(0)$. If $g(z) = e^{e^z} + 1$, then $g'(g-1)^n$ does not assume zero.

Theorem 16 [15]　Let $n \geqslant 2$ and $Q_d(z,f)$ be a differential polynomial in f of degree d with rational functions as its coefficients. Suppose that p is a nonzero rational function, α is a nonconstant polynomial and $d \leqslant n-2$. If the following differential equation

$$f^n + Q_d(z,f) = p(z)e^{\alpha(z)}$$

admits a meromorphic function f with finitely many poles, then f has the following form $f(z) = q(z)e^{r(z)}$ and $Q_d(z,f) \equiv 0$, where $q(z)$ is a rational function and $r(z)$ is a polynomial with $q^n = p$, $nr(z) = \alpha(z)$. In particular, if p is a polynomial, then q is a polynomial, too.

Remark 6　The condition $d \leqslant n-1$ is necessary. For example, $f(z) = e^z + z$ solve the following differential equation

$$ff'^2 - 2zf'^2 - z^2 f' = e^{3z},$$

where $n = d = 2$.

In 1959, Hayman [29] studied Picard-value problem of some type of differential polynomial of transcendental entire functions. In fact, he proved the following results.

Theorem 17　Let f be a non-constant meromorphic function in $\mathbf{C}$, a, $b(\neq 0)$ be finite values, and k be a positive integer. Then either $f = a$ or $f^{(k)} = b$ has at least one root. Moreover, if f is transcendental, then either $f = a$ or $f^{(k)} = b$ has infinitely many roots.

As a consequence of Theorem 17, it is easy to prove

Theorem 18　If f be a transcendental entire function, then $f^2 + af'$ has infinitely many zeros for each finite nonzero complex number a.

In fact, if f is an entire function, then $g = 1/f$ has no zero. It follows from Theorem 17 that $g' - 1/a$ has infinitely many zeros, so does $f^2 + af'$.

Motivated by this result, we research the above nonlinear differential equation first. With this in hand, we generalize Theorem 18 to a certain kind of differential polynomials.

Theorem 19 [15]　Let f be a transcendental entire function,

$$Q_n(f) = a_nf^n + \cdots + a_1f + a_0$$

a polynomial of degree $n \geqslant 3$. Then $f' + Q_n(f)$ assumes every complex number α infinitely many times, except a possible value $Q_n(-a_{n-1}/(na_n))$. On the other hand, if $f' + Q_n(f)$ assumes the complex value $Q_n(-a_{n-1}/(na_n))$ finitely many times, then

$$Q_n(z) = a_n(z + a_{n-1}/(na_n))^n + B(z + a_{n-1}/(na_n)) + Q_n(-a_{n-1}/(na_n)),$$

$$f(z) = A\mathrm{e}^{-Bz} - a_{n-1}/(na_n).$$

From Theorem 19, we have

Corollary 4 Let f be a transcendental entire function, then $f' + a_nf^n + P_d(f)$ assumes every finite complex number α infinitely many times, where $n \geqslant 3$, $a_n \neq 0$ and $P_d(f)$ is a nonlinear polynomial of f with degree $d \leqslant n - 2$ or a constant.

Remark 7 In Theorem 19 and Corollary 4, the restrict condition $n \geqslant 3$ is necessary. For example, let $f = \mathrm{e}^z - 1$, then

$$f' + f^2/2 = \mathrm{e}^{2z}/2 - 1/2$$

does not assume 1/2. In Corollary 4, the restrict condition $d \leqslant n - 2$ is necessary. For example, let $f = \mathrm{e}^z + 1$, then $f' + f^3/3 - f^2$ does not assume $-2/3$. The restrict condition $P_d(f)$ is a nonlinear polynomial is necessary. For example, $f(z) = \mathrm{e}^z$, then $f' - 2f$ does not assume 0, Theorem 19 and Corollary 4 are not valid for meromorphic functions. For example, $f(z) = \tan z$, then $f' + f^4 + f^2 = \sec^4 z$ does not assume 0.

Corollary 5 Let $\mathscr{F}$ be a family of holomorphic functions in domain D,

$$Q_n(z) = a_nz^n + \cdots + a_1z + a_0$$

a polynomial of degree $n \geqslant 3$. If for every function $f \in \mathscr{F}$, $f' + Q_n(f)$ does not assumes a complex number $a \neq Q_n(-a_{n-1}/(na_n))$ in D, then $\mathscr{F}$ is normal in D.

Corollary 6 Let $\mathscr{F}$ be a family of holomorphic functions in domain D. If for every function $f \in \mathscr{F}$, $f' + a_nf^n + P_d(f)$ does not assumes a complex number a in D, where $n \geqslant 3$, $a_n \neq 0$ and $P_d(f)$ is a nonlinear polynomial of f with degree $d \leqslant n - 2$ or a constant, then $\mathscr{F}$ is normal in D.

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非线性复微分方程研究的新进展

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摘要: 利用 Nevanlinna 理论和 Wiman-Valiron 理论,研究了代数微分方程没有允许解的问题,给出了几类非线性微分方程整函数解的结构,并利用这些结果将 Hayman 定理推广到微分多项式,综述了在非线性复微分方程及其应用研究中的最新进展.

关键词: Nevanlinna 值分布理论; 非线性微分方程; 微分多项式; 亚纯函数解; 整函数解

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