文章编号:1000-5862(2016)06-0578-09

Normal Families of Meromorphic Functions Concerning Composite Meromorphic Functions and Fixed Points

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Abstract: Then following result is proved: If f is a transcendental entire function, such that the family G of all functions g holomorphic in the domain $D\subset\mathbf{C}$ for which every zero of $f(g)-\alpha_1$ is of multiplicity $\geqslant 2$, $f(g)-\alpha_2$ and $g-\alpha_2$ share 0 IM in D, where α_1 and α_2 are two distinct finite values, then G is normal in D. This result extends Theorem 1 of paper in Bergweiler. The following result is also proved: If R is a rational function with $\deg R\geqslant 2$ (respectively $\geqslant 3$, and R has three distinct finite fixed points in the complex plane) such that the family $\mathscr{F}$ of all functions f holomorphic (respectively meromorphic) in the domain $D\subset\mathbf{C}$ for which $R\circ f(z)-z$ and $f(z)-z$ share 0 IM in D, then $\mathscr{F}$ is normal in D. The results extend the corresponding results due to Fang-Yuan and Chang-Fang-Zalcman respectively. Examples are provided to show that the main results in this paper, in a sense, are the best possible.

Key words: meromorphic functions; composite functions; fixed-points; shared values; normal criterions

中图分类号: O 174.52 **文献标志码:** A **DOI:** 10.16357/j.cnki.issn1000-5862.2016.06.07

0 Introduction and Main Results

Let D be a domain on the complex plane $\mathbf{C}$, and let $\mathscr{F}$ be a family of meromorphic functions D. The family $\mathscr{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\}\subset\mathscr{F}$ contains either a subsequence that converges to a meromorphic function uniformly on each compact subset of D, or a subsequence which converges to ∞ uniformly on each compact subset of D, see, e.g., Hayman [1], Schiff [2] and Yang [3].

Let f and g be two nonconstant meromorphic functions in the complex plane, and let a be a value in the extended plane. We say that f and g share the value a CM in the complex plane, provided that f and g have the same a-points in the complex plane and each common a-point of f and g has the same multiplicity related to f and g. We say that f and g share the value a IM in the complex plane, provided that f and g have the same a-points and each common a-point of f and g is counted only once. We say that f and g share a CM in D, provided that f and g have the same a-points in D, and each such common a-point of f and g has the same multiplicity related to f and g. We say that f and g share a IM in D, provided that f and g have the same a-points in D, and each such common a-point of f and g is counted only once [4]. Throughout this paper, we denote by $\mu(f)$, $\rho(f)$ and $\lambda(f)$ the lower order of f, the order of f and the exponent of convergence of zeros of f respectively [1,4-5]. If $\mu(f)=\rho(f)$, we say that f is of regular growth. The iterates $f^n: D_n\to\mathbf{C}$ of a holomorphic function $f: D\to\mathbf{C}$ are defined as $D_1:=D$, $f^1:=f$ and $D_n:=f^{-1}(D_{n-1})$, $f^n:=f^{n-1}\circ f$ for $n\in\mathbf{N}$, $n\geqslant 2$. Note that $D_2=f^{-1}(D_1)\subset D=D_1$ and thus $D_{n+1}\subset D_n\subset D$ for all $n\in\mathbf{N}$, which can be found, e.g., in Ess´en and Wu [6] and Bergweiler [7]. Let D be a domain in the complex plane and let f be holomorphic in D. We say that f^k has a fixed point $a\in D$ if the following conditions hold: $f^j(a)\in D$ for $1\leqslant j\leqslant k$, and $f^k(a)=a$. Let $R(z)=P_1(z)/P_2(z)$ be a noncon-

收稿日期:2016-10-10
基金项目:国家自然科学基金(11171184,11461042)和山东省自然科学基金(Z2008A01,ZR2014AM011)资助项目.
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stant rational function, where P_1 and P_2 are two relatively prime polynomials. Then, we call $\max\{\deg P_1, \deg P_2\} =: \deg R$ the degree of R. We recall the following results proved by Rosenbloom[8] in 1952.

Theorem 1 Let f be a transcendental entire function and let $k \in \mathbf{N}, k \geqslant 2$. Then the k-th iterate f^k has infinitely many fixed points.

Theorem 2 Let P be a polynomial with $\deg P \geqslant 2$, and let f be a transcendental entire function. Then the composite function $P(f)$ has infinitely many fixed points.

Later on, Gross-Osgood[9] proved the following result which extends Theorem 2.

Theorem 3 Let R be a rational function with $\deg R \geqslant 3$, and let f be a transcendental meromorphic function. Then the composite function $R(f)$ has infinitely many fixed points.

In 1998, Ess´en-Wu[6] proved the following normality criterion corresponding to Theorem 1, thereby answering a question of Yang[10].

Theorem 4[6] Let D be a domain in the complex plane, let $\mathscr{F}$ be a family of holomorphic functions in a domain $D \subset \mathbf{C}$. If for every $f \in \mathscr{F}$, there exists $k = k(f)$ such that f^k has no fixed points in D, then $\mathscr{F}$ is a normal family.

Fang-Yuan[11] and Chang-Fang-Zalcman[12] proved the following results respectively, which are corresponding to Theorems 2 and Theorem 3.

Theorem 5[11] Let $\mathscr{F}$ be a family of holomorphic functions defined on a domain D, and let P be a polynomial with degree $\deg P \geqslant 2$. If, for any $f \in \mathscr{F}$, the composite function $P(f)$ has no fixed-point, then $\mathscr{F}$ is normal in D.

Theorem 6[12] Let $\mathscr{F}$ be a family of meromorphic functions defined on a domain D, and let R be a rational function with $\deg R \geqslant 3$. If, for any $f \in \mathscr{F}$, the composite function $R(f)$ has no fixed point in D, then $\mathscr{F}$ is normal in D.

Regarding Theorems 5 and Theorem 6, we propose the following questions:

Question 1 Let D be a domain in the complex plane, let $\mathscr{F}$ ($\mathscr{H}$, respectively) be a family of meromorphic functions (entire functions, respectively.), and let R (P, respectively) be a rational function (a polynomial, respectively) with degree $\deg R \geqslant 3$ ($\deg P \geqslant 2$, respectively). If, for every $g \in \mathscr{F}$ ($h \in \mathscr{H}$, respectively) both g (h, respectively) and the composite function $R(g)$ ($P(h)$, respectively) have the same fixed points in D, or both g (h, respectively) and the composite function $R(g)$ ($P(h)$, respectively) share a finite nonzero complex number a in D, is $\mathscr{F}$ ($\mathscr{H}$, respectively) normal in D?

Question 2 Let D be a domain in the complex plane, let $\mathscr{F}$ be a family of meromorphic functions, and let f be a transcendental meromorphic function. If, for every $g \in \mathscr{F}$, both g and the composite function $f(g)$ have the same fixed points in D, or both g and the composite function $f(g)$ share a finite nonzero complex number a in D, is $\mathscr{F}$ normal in D?

We will prove the following results to deal with Questions 1 and Question 2, where Theorem 7 extends Theorem 1 in Bergweiler[13], Theorem 8 and Theorem 10 extend the corresponding results in Fang-Yuan[11] and Chang-Fang-Zalcman[12] respectively.

Theorem 7 Let $\mathscr{F}$ be a family of holomorphic functions defined on a domain D, and let f be a transcendental entire function. Suppose that α_1 and α_2 are two distinct finite values. If, for any $g \in \mathscr{F}$, every zero of $f(g) - \alpha_1$ is of multiplicity $\geqslant 2$, g and $f(g)$ share α_2 IM in D, then $\mathscr{F}$ is normal in D.

From Theorem 7 we can get the following result.

Corollary 1 Let $\mathscr{F}$ be a family of holomorphic functions defined on a domain D, and let f be a transcendental entire function. If, for any $g \in \mathscr{F}$, there exist two distinct finite values α_1 and α_2 such that every zero of $f(g) - \alpha_1$ and $f(g) - \alpha_2$ is of multiplicity $\geqslant 2$, and that $f(g)$ and g have the same fixed points in D, then $\mathscr{F}$ is normal in D.

The following example shows that the assumption "for any $g \in \mathscr{F}$, every zero of $f(g) - \alpha_1$ is of multiplicity $\geqslant 2$" in Theorem 7 is necessary.

Example 1 Take $f(z) = 1 + (1 - z)e^z$, $\mathscr{F} = \{g_n\}$, where $g_n(z) = \sin^2(nz)$, and take $\alpha = 1$. Then, we can find that for any $g_n \in \mathscr{F}$, g_n and $f(g_n)$ share 1 CM in D, and that not all the zeros of $f(g_n)$ is of multiplicity $\geqslant 2$ in the domain $D = \{z: |z| < \varepsilon\}$, where $\varepsilon > 0$ is any fixed positive number. But $\mathscr{F}$ is not normal at $z_0 = 0$.

We also prove the following result.

Theorem 8 Let $\mathscr{F}$ be a family of holomorphic functions defined on a domain D, let R be a rational function with deg $R\geqslant 2$, and let R have at least two distinct fixed points in the complex plane. Suppose that $\alpha\not\equiv 0$ is a nonconstant holomorphic function defined on a domain D. If, for any $f\in\mathscr{F}$, $f-\alpha$ and $R(f)-\alpha$ share 0 IM in D, then $\mathscr{F}$ is normal in D.

From Theorem 8 we can get the following result.

Corollary 2 Let $\mathscr{F}$ be a family of holomorphic functions defined on a domain D, let R be a rational function with deg $R\geqslant 2$, and let R have at least two distinct fixed points in the complex plane. If, for any $f\in\mathscr{F}$, f and $R(f)$ have the same fixed points in D, then $\mathscr{F}$ is normal in D.

Proceeding as in the proof of Theorem 8 in Section 2, we can get the following result.

Theorem 9 Let $\mathscr{F}$ be a family of holomorphic functions defined on a domain D, let R be a rational function with deg $R\geqslant 2$, and let R have at least two distinct fixed points in the complex plane. Suppose that $\alpha\neq 0$ is a finite value in the complex plane. If, for any $f\in\mathscr{F}$, f and $R(f)$ share α IM in D, then $\mathscr{F}$ is normal in D.

For a family of meromorphic functions defined on a domain D we will prove the following result which is corresponding to Theorem 6 and deals with Question 1.

Theorem 10 Let $\mathscr{F}$ be a family of meromorphic functions defined on a domain D, and let $R=P_1/P_2$ be a nonconstant rational function, where P_1 and P_2 are two relatively prime polynomials such that deg $R=\max\{\deg P_1,\deg P_2\}\geqslant 3$ and such that R has at least three distinct finite fixed points in the complex plane. If, for any $f\in\mathscr{F}$, $f-z$ and $R(f)-z$ share 0 IM, then $\mathscr{F}$ is normal in D.

We give the following two examples.

Example 2 [11] Take $P(z)=z$, $f_n(z)=z+e^{nz}$, $D=\{z:|z|<1\}$. Then it is easy to see that $f_n(z)-z\neq 0$ and $P(f_n(z))-z\neq 0$ in D, and that the family of holomorphic functions $\{f_n(z)\}$ is not normal in D. This example shows that the assumption "deg $R\geqslant 2$" in Theorem 8 is necessary.

Example 3 [14] Let $R(z)=(z^2+1)/(z^2-1)$, let

$$f(z)=\frac{\sin\sqrt{z}}{\sqrt{z}\cos\sqrt{z}}=\frac{\sum\limits_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}z^j}{\sum\limits_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}z^j},$$

$z\in\{z:|z-1|<\delta\}=:D_\delta$, and let $\psi(z)=(1+z)^{1/2}$. Set $\mathscr{F}=\{f_n\}$, where $f_n(z)=\mathrm{i}n^{1/2}\psi(z)f(n(z-1))$, $n=1,2,\cdots$. Then, for sufficiently small positive number δ, we can find that $f_n(z)\neq z$ and

$$R(f_n(z))=z-\frac{z+1}{1-2n\left(\frac{\sin\sqrt{n(z-1)}}{\sqrt{n(z-1)}}\right)^2}\neq z$$

on D_δ, and so $f_n(z)-z$ and $R(f_n(z))-z$ share 0 IM in D_δ. Noting that for any $f_n\in\mathscr{F}$, f_n has poles and zeros in $|z-1|<\varepsilon$, where $\varepsilon>0$ is a given sufficiently small positive number, we can see that $\mathscr{F}$ is not equicontinuous at $z=1$, and so $\mathscr{F}$ is not normal in D_δ. This example shows that the assumption "deg $R\geqslant 3$" in Theorem 10 is necessary.

1 Preliminaries

In this section, we introduce some important lemmas to prove the main results in this paper. First of all, we introduce the following result due to Pang-Zalcman, which plays an important role in studying the theory of normal families of meromorphic functions.

Lemma 1 [15-17] (Pang-Zalcman Lemma) Let $\mathscr{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multi-plicities at least k, and suppose that there exists $A\geqslant 1$ such that $|f^{k}(z)|\leqslant A$ whenever $f(z)=0$, $f\in\mathscr{F}$. Then, if $\mathscr{F}$ is not normal, then, for each $-1<\alpha\leqslant k$, we have:

(i) a number $0<r<1$;

(ii) points z_n, $|z_n|<r$;

(iii) functions $f_n\in\mathscr{F}$;

(iv) positive numbers $\rho_n\to 0$ such that

$$f(z_n+\rho_n\zeta)/\rho_n^{\alpha}=:g_n(\zeta)\to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$ such that $g^{\#}(\zeta)\leqslant g^{\#}(0)=kA+1$.

Remark 1 Suppose additionally in Lemma 1 that $\mathscr{F}$ is a family of zero-free meromorphic functions in the domain D. Then the real number α in Lemma 1 can be such that $-1<\alpha<\infty$

We also need the following result to prove the main results in this paper.

Lemma 2 [18] Let f be a meromorphic function on $\mathbf{C}$. If f has bounded spherical derivative on $\mathbf{C}$, then f is of order at most 2. If, in addition, f is entire, then the order of f is at most 1.

Next we introduce the following result which is originally due to Clunie[19] for the composition of two transcendental entire functions. Later on, this result was proved, e. g., in Yang–Yi[4] for the composition of a nonconstant entire function and a transcendental meromorphic function as follows.

Lemma 3[4,19] Suppose that g is a transcendental meromorphic function and that h is a nonconstant entire function. Then

$$\lim_{r\to\infty} T(r,g(h))/T(r,h) = \infty$$

The next result is due to Clunie[19].

Lemma 4 If f and g are entire functions, then

$$M(r,f\circ g) = M((1+o(1))M(r,g),f),$$

as $r \notin E$ and $r\to\infty$, where and what follows, E denotes an exceptional set of finite logarithmic measure, not necessarily the same at each occurrence.

Finally we give the following result due to Bergweiler[13], which plays an important role in proving the main results of this paper.

Lemma 5 Let $f: \mathbf{C}\to\mathbf{C}\cup\{\infty\}$ be a rational function of degree at least 2, let $D\subset\mathbf{C}$ be a domain, let $h: D\to\mathbf{C}\cup\{\infty\}$ be meromorphic and nonconstant and Let Γ be a family of all holomorphic functions $g: D\subset\mathbf{C}$ such that $f(g(z))\neq h(z)$ for all $z\in D$. then Γ is normal.

Lemma 6[20] Suppose that f and g are two nonconstant rational functions such that f and g share $0, 1, \infty$ CM. Then $f=g$.

2 Proofs of the Theorems

Proof of Theorem 7 We may assume that $D = \{z: |z|<1\}$. Suppose that $\mathscr{F}$ is not normal in D. Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 1 and the assumptions of Theorem 7 we can find that there exist points $z_n\to 0$, $|z_n|<1$, positive numbers $\rho_n, \rho_n\to 0^+$ and a subsequence of functions $g_n\in\mathscr{F}$ such that

$$g_n(z_n+\rho_n\zeta) =: h_n(\zeta)\to h(\zeta), \tag{1}$$

and so

$$f(g_n(z_n+\rho_n\zeta)) = f(h_n(\zeta))\to f(h(\zeta)),$$

spherical uniformly on compact subsets of $\mathbf{C}$, where h is some nonconstant entire function. From (1), Lemma 2 and Marty's Theorem we have $\rho(h)\leqslant 1$. By the assumptions of Theorem 7 and Hurwitz's Theorem we deduce that every zero of $f(h)-\alpha_1$ is of multiplicity $\geqslant 2$, and that $f(h)$ and h share α_2 IM in the complex plane. Therefore, by the second fundamental theorem we have

$$T(r,f(h)) \leqslant \overline{N}\left(r,\frac{1}{f(h)-\alpha_1}\right) + \overline{N}\left(r,\frac{1}{f(h)-\alpha_2}\right) + S(r,f(h)) \leqslant \frac{1}{2}N\left(r,\frac{1}{f(h)-\alpha_1}\right) + \overline{N}\left(r,\frac{1}{f(h)-\alpha_2}\right) + S(r,f(h)) \leqslant \frac{1}{2}T(r,f(h)) + T(r,h) + S(r,f(h)),$$

which implies

$$T(r,f(h)) \leqslant 2T(r,h) + S(r,f(h)),$$

and so

$$\lim_{r\to\infty, r\notin E} T(r,f(h))/T(r,h) \leqslant 2. \tag{2}$$

On the other hand, by Lemma 3 we have

$$\lim_{r\to\infty} T(r,f(h))/T(r,h) = \infty \tag{3}$$

From (2) and (3) we get a contradiction. This completes the proof of Theorem 7.

Proof of Corollary 1 We may assume that $D = \{z: |z|<1\}$. Suppose that $\mathscr{F}$ is not normal in D. Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 1 and the assumptions of Corollary 1 we can find that there exist points $z_n\to 0$, $|z_n|<1$, positive numbers $\rho_n, \rho_n\to 0^+$ and a subsequence of functions $g_n\in\mathscr{F}$ such that (1) holds, and so

$$g_n(z_n+\rho_n\zeta) - (z_n+\rho_n\zeta) = h_n(\zeta) - (z_n+\rho_n\zeta)\to h(\zeta)$$

and

$$f(g_n(z_n+\rho_n\zeta)) - (z_n+\rho_n\zeta) = f(h_n(\zeta)) - (z_n+\rho_n\zeta)\to f(h(\zeta)),$$

spherical uniformly on compact subsets of $\mathbf{C}$, where h is some nonconstant entire function. From (1), Lemma 2 and Marty's Theorem we have $\rho(h)\leqslant 1$. By Hurwitz's Theorem and the assumption that for any $g\in\mathscr{F}$, $f(g)$ and g have the same fixed points we deduce that $f(h)$ and h share 0 IM. From Lemma 3, Hurwitz's Theorem and the assumption that for any $g\in\mathscr{F}$, every zero of $f(g)-\alpha_1$ and $f(g)-\alpha_2$ is of multiplicity $\geqslant 2$. We consider the following two cases:

(i) Suppose that $\alpha_1\neq 0$ and $\alpha_2\neq 0$. Then, by the second fundamental theorem we have

$$2T(r,f(h)) \leqslant \overline{N}\left(r,\frac{1}{f(h)}\right) + \overline{N}\left(r,\frac{1}{f(h)-\alpha_1}\right) + \overline{N}\left(r,\frac{1}{f(h)-\alpha_2}\right) + S(r,f(h)) \leqslant \overline{N}\left(r,\frac{1}{h}\right) + \frac{1}{2}N\left(r,\frac{1}{f(h)-\alpha_1}\right) + \frac{1}{2}N\left(r,\frac{1}{f(h)-\alpha_2}\right) +$$

$S(r,f(h)) \leqslant T(r,f(h)) + T(r,h) + S(r,f(h))$,

and so from Lemma 3 we have $T(r,f(h)) \leqslant T(r,h) + S(r,f(h)) \leqslant S(r,f(h))$, which is impossible.

(ii) Suppose that $0 \in \{\alpha_1,\alpha_2\}$, say $\alpha_2 = 0$ and $\alpha_1 \neq 0$. Then, by Theorem 7 we can get the conclusion of Corollary 1.

This completes the proof of Corollary 1.

Proof of Theorem 8 Suppose that $z_0 \in D$ is any given point. We consider the following three cases:

(i) Suppose that $z_0 \in D$ is such a point that for any given sequence of functions $f_n \in \mathscr{F}$, there exist a subsequence of f_n, say itself such that $f_n(z) \neq \alpha(z)$ for all $z \in \Delta(z_0,\delta) \subset D$. Combining this with the assumptions of Theorem 8, we can find that $f_n(z) - \alpha(z)$ and $R(f_n(z)) - \alpha(z)$ share 0 IM in $\Delta(z_0,\delta)$, and so $R(f_n(z)) - \alpha(z) \neq 0$ for all $z \in \Delta(z_0,\delta) \subset D$. This together with Lemma 5 implies that $\{f_n\}$ is normal in $\Delta(z_0,\delta)$.

(ii) Suppose that $z_0 \in D$ is such a point that there exist an infinite subsequence of $f_n \in \mathscr{F}$ such that $f_n(z_n) = \alpha(z_n)$, where z_n is an infinite sequence of points such that $z_n \to z_0$. Then, by the assumptions of Theorem 8 we have $R(f_n(z_n)) = \alpha(z_n)$, and so we have

$$R(\alpha(z_n)) = \alpha(z_n). \tag{4}$$

By the assumptions of Theorem 8 we know that there exist $l \geqslant 2$ distinct finite complex values $\hat{z}_1,\hat{z}_2,\cdots,\hat{z}_l$ such that $R(\hat{z}_j) = \hat{z}_j$ for $1 \leqslant j \leqslant l$, where $l \geqslant 2$ is a positive integer satisfying $2 \leqslant l \leqslant n$. Noting that α is a holomorphic function in D, we have $\alpha(z_0) \neq \infty$. We consider the following two subcases:

Subcase 1 Suppose that $\alpha(z_0) \in \{\hat{z}_j: 1 \leqslant j \leqslant l\}$. Then, we can deduce by (4) that there exist a subsequence of points z_n, say itself such that $\alpha(z_n) = \hat{z}_{j0}$ for some point $\hat{z}_{j0} \in \{\hat{z}_1,\hat{z}_2,\cdots,\hat{z}_l\}$. Combining this with the isolation of zeroes of a nonconstant analytic function in a domain and the supposition $\lim\limits_{n\to\infty} z_n = z_0$, we can deduce that α is a constant, which contradicts the assumption of Theorem 8.

Subcase 2 Suppose that $\alpha(z_0) \notin \{\hat{z}_j: 1 \leqslant j \leqslant l\}$. By letting $n \to \infty$ we have from (4) that $R(\alpha(z_0)) = \alpha(z_0)$, and so $\alpha(z_0) \in \{\hat{z}_j: 1 \leqslant j \leqslant l\}$, which is impossible.

(iii) Suppose that $z_0 \in D$ is such a point that there exist an infinite sequence of functions $f_n \in \mathscr{F}$ such that $f_n(z_0) = \alpha(z_0)$ and such that $f_n(z) \neq \alpha(z)$ for all $z \in \Delta'(z_0,\delta)$, where δ is some positive number. Combining this with the assumption that $R(f_n) - \alpha$ and $f_n - \alpha$ share 0 IM, we can get $R(f_n)(z) - \alpha(z) \neq 0$ for all $z \in \Delta'(z_0,\delta)$, and so by Lemma 5 we can find that $\{f_n\}$ is normal in $\Delta'(z_0,\delta)$, and so there exist a subsequence of $\{f_n\}$, say itself such that

$$f_n(z) \to f(z) \tag{5}$$

or

$$f_n(z) \to \infty, \tag{6}$$

spherical uniformly on compact subsets of $\Delta'(z_0,\delta)$, where $f \not\equiv \infty$ is a holomorphic function defined on $\Delta'(z_0,\delta)$. We discuss the following two subcases:

Subcase 1 Suppose that (6) holds. Then, by (6) and the Cauchy's Integral Formula we can get

$$(1/f_n)' \to 0, \tag{7}$$

spherical uniformly on compact subsets of $\Delta'(z_0,\delta)$. Next we denote by Z_h and P_h the numbers of zeros and poles of a meromorphic function h in $\Delta(z_0,\delta)$ respectively, where each zero and each pole are counted according their multiplicities. Then, by (6) and (7), the argument principle and the assumption that $\alpha(z) \not\equiv 0$ for any $z \in \Delta(z_0,\delta)$ we get

$$Z_{1/f_n-1/\alpha} - P_{1/f_n-1/\alpha} = \frac{1}{2\pi i}\int_{|z-z_0|=\delta}\frac{(1/f_n(z) - 1/\alpha(z))'}{1/f_n(z) - 1/\alpha(z)}dz \to -\frac{1}{2\pi i}\int_{|z-z_0|=\delta}\frac{\alpha'(z)}{\alpha(z)}dz = -n(z_0,1/\alpha), \tag{8}$$

where the notation $n(z_0,1/\alpha)$ is such that $n(z_0,1/\alpha) = 0$ if $\alpha(z_0) \neq 0$, and that $n(z_0,1/\alpha)$ is the multiplicity of the zero $z = z_0$ of α if $\alpha(z_0) = 0$. We consider the following three subcases:

Subcase 1.1 Suppose that $\alpha(z_0) \neq 0$ and that there exist a subsequence of $\{f_n\}$, say itself such that for every positive integers $n \geqslant 1$, we have $f_n(z_0) - \alpha(z_0) \neq 0$. Combining this with (6) and (8), we have $Z_{f_n-\alpha} = P_{f_n} = 0$ and so $1/(f_n - \alpha)$ is a holomorphic function in $\Delta(z_0,\delta)$ for sufficiently large positive integer n. Moreover, from (6) we have

$$1/(f_n - \alpha) \to 0, \tag{9}$$

spherical uniformly on compact subsets of $\Delta'(z_0,\delta)$. By (9) and the Maximum Modulus Theorem we can find that there exists some positive constant $M(\delta)$ depending only upon δ such that for sufficiently large positive integer n we have

$$|1/(f_n-\alpha)|\leq M(\delta) \tag{10}$$

for all $z\in\bar{\Delta}(z_0,\delta/2)$. From (10) we can find that $\{1/(f_n-\alpha)\}$, and so f_n is normal in $\Delta(z_0,\delta/2)$.

Subcase 1.2 Suppose that $\alpha(z_0)=0$ and that there exist a subsequence of $\{f_n\}$, say itself such that for every positive integers $n\geq 1$, we have $f_n(z_0)-\alpha(z_0)=f_n(z_0)\neq 0$. Combining this with (6), we can find that $1/f_n$ is holomorphic in $\Delta(z_0,\delta)$ for sufficiently large positive integer. Therefore, from (6) and the Maximum Modulus Theorem we can see that there exists some positive number $M(\delta)$ depending only upon δ such that for sufficiently large positive integer n we have

$$|1/f_n(z)|\leq M(\delta) \tag{11}$$

for all $z\in\bar{\Delta}(z_0,\delta/2)$. From (11) we can see that $\{1/f_n\}$, and so $\{f_n\}$ is normal in $\Delta(z_0,\delta/2)$.

Subcase 1.3 Suppose that there exist a subsequence of $\{f_n\}$, say itself such that for every positive integers $n\geq 1$, we have $f_n(z_0)-\alpha(z_0)=0$. Combining this with the assumption that $R(f_n)-\alpha$ and $f_n-\alpha$ share 0 IM, we have

$$R(\alpha(z_0))=\alpha(z_0).$$

On the other hand, by the assumptions of Theorem 8 we can see that there exists a value α_1 such that $R(\alpha_1)=\alpha_1$ and $\alpha_1\neq\alpha(z_0)$. This together with the fact $R(z)\neq z$ for all $z\in\Delta'(z_0,\delta)$ implies that $1/(f_n-\alpha_1)$ is a holomorphic function in $\Delta(z_0,\delta)$ for sufficiently large positive integer n. Moreover, from (5) we have

$$1/(f_n-\alpha_1)\to 1/(f-\alpha_1), \tag{12}$$

spherical uniformly on compact subsets of $\Delta'(z_0,\delta)$. From (12) and Hurwitz's Theorem we can see that $1/(f-\alpha_1)$ is a holomorphic function in $\Delta'(z_0,\delta)$. Therefore, by the Maximum Modulus Theorem we can see that there exists some positive number $M(z_0,\delta)$ depending only upon f and δ such that for sufficiently large positive integer n we have

$$|1/(f_n-\alpha_1)|\leq M(z_0,\delta) \tag{13}$$

for all $z\in\bar{\Delta}(z_0,\delta/2)$. From (13) we can see that $\{1/(f_n-\alpha_1)\}$, and so $\{f_n\}$ is normal in $\Delta(z_0,\delta/2)$.

Subcase 2 Suppose that (5) holds and that $f\not\equiv 0$ in $\Delta'(z_0,\delta)$. Then, by Hurwitz's Theorem and the assumption that $\{f_n\}$ is a family of holomorphic functions we have that f is a holomorphic function in $\Delta'(z_0,\delta)$. Therefore, by the Maximum Modulus Theorem we can find that there exists some positive constant $M(\delta)$ depending only upon δ such that for sufficiently large positive integer n we have

$$|f_n(z)|\leq M(\delta)$$

for all $z\in\bar{\Delta}(z_0,\delta/2)$. This together with the fact that $\{f_n\}$ is a family of holomorphic functions in $\Delta(z_0,\delta)$ implies that $\{f_n\}$ is normal in $\Delta(z_0,\delta/2)$, this reveals the conclusion of Theorem 8.

Suppose that $f(z)=0$ for every $z\in\Delta'(z_0,\delta)$. Then, by the Maximum Modulus Theorem we can find a constant $M>0$ such that for sufficiently large positive integer n we have $f_n(z)\leq M$ for all $z\in\Delta(z_0,\delta)$, which also reveals the conclusion of Theorem 8.

Theorem 8 is thus completely proved.

Proof of Theorem 10 Suppose that $z_0\in D$ is any given point. We consider the following three cases:

(i) Suppose that $z_0\in D$ is such a point that for a given sequence of functions $f_n\in\mathscr{F}$, there exist a subsequence of f_n, say itself such that $f_n(z)\neq z$ for all $z\in\Delta(z_0,\delta)\subset D$, where and in what follows, $\Delta(z_0,\delta)=\{z: |z-z_0|<\delta\}$, $\delta>0$ is some positive number. Then, by the assumptions of Theorem 10 we can find that $f_n(z)-z$ and $R(f_n(z))-z$ share 0 IM in $\Delta(z_0,\delta)$, and so $R(f_n(z))-z\neq 0$ for all $z\in\Delta(z_0,\delta)\subset D$. This together with Theorem 6 implies that $\{f_n\}\subset\mathscr{F}$ is normal in $\Delta(z_0,\delta)$.

(ii) Suppose that $z_0\in D$ is such a point that for a given sequence of functions $f_n\in\mathscr{F}$, there exist an infinite subsequence of f_n, say itself such that $f_n(z_n)=z_n$, where z_n is a sequence of points such that $z_n\to z_0$. Then, by the assumptions of Theorem 10 we have $R(f_n(z_n))=z_n$, and so we have

$$R(z_n)=z_n. \tag{14}$$

Noting that there are at least $l\leq n$ distinct finite complex values $\hat{z}_1,\hat{z}_2,\cdots,\hat{z}_l$ such that $R(\hat{z}_j)=\hat{z}_j$ for $1\leq j\leq l$, we consider the following two subcases:

Subcase 1 Suppose that $\hat{z}_{j0}\in\{\hat{z}_j: 1\leq j\leq l\}$. Then, we can deduce by (14) that there exist a subsequence of points z_n, say itself such that $z_n=\hat{z}_{j0}$ for some point $\hat{z}_{j0}\in\{\hat{z}_1,\hat{z}_2,\cdots,\hat{z}_l\}$. Combining this with the isolation of zeroes of a nonconstant analytic function in a domain and the supposition $\lim_{n\to\infty}z_n=z_0$, we can deduce that z is a constant, which is impossible.

Subcase 2 Suppose that $z_0 \notin \{\hat{z}_j : 1 \leqslant j \leqslant l\}$. By letting $n \to \infty$ we have from (14) that $R(z_0) = z_0$, and so $z_0 \in \{\hat{z}_j : 1 \leqslant j \leqslant l\}$, which is impossible.

(iii) Suppose that $z_0 \in D$ is such a point that for a given sequence of functions $f_n \in \mathscr{F}$, there exist an infinite subsequence of f_n, say itself such that $f_n(z_0) = z_0$ and such that $f_n(z) \neq z$ for all $z \in \Delta'(z_0, \delta)$, where δ is some positive number. Then, by Theorem 6 we can find that $\{f_n\}$ is normal in $\Delta'(z_0, \delta)$, therefore, there exist a subsequence of $\{f_n\}$, say itself such that

$$f_n \to f, \tag{15}$$

or

$$f_n \to \infty, \tag{16}$$

spherical uniformly on compact subsets of $\Delta'(z_0, \delta)$, where $f \not\equiv \infty$ is a meromorphic function defined in $\Delta'(z_0, \delta)$. We consider the following three subcases:

Subcase 1 Suppose that (16) holds. Then, by the Cauchy's Integral Formula we can get

$$(1/f_n)' \to 0,$$

spherical uniformly on compact subsets of $\Delta'(z_0, \delta)$. Next we denote by Z_h and P_h the numbers of zeros and poles of a meromorphic function h in $\Delta(z_0, 3\delta/4)$ respectively, where each zero and each pole are counted according their multiplicities respectively. Then, by (15) ~ (16) and the argument principle we get

$$Z_{1/f_n - 1/z} - P_{1/f_n - 1/z} = \frac{1}{2\pi i}\int_{|z-z_0| = 3\delta/4} \frac{(1/f_n(z) - 1/z)'}{1/f_n(z) - 1/z}dz \to -\frac{1}{2\pi i}\int_{|z-z_0| = 3\delta/4} \frac{1}{z}dz. \tag{17}$$

We consider the following three subcases:

Subcase 1.1 Suppose that $z_0 \neq 0$ and that there exist a subsequence of $\{f_n\}$, say itself such that for every positive integers $n \geqslant 1$, we have $f_n(z_0) - z_0 \neq 0$. Then $\frac{1}{2\pi i}\int_{|z-z_0|=3\delta/4} \frac{1}{z}dz = 0$. Combining this with (16) and (17), we have $Z_{f_n - z} = P_{f_n} = 0$, and so $1/(f_n - z)$ is a holomorphic function in $\Delta(z_0, 3\delta/4)$ for sufficiently large positive integer n. Moreover, by (16) we have

$$1/(f_n - z) \to 0, \tag{18}$$

spherical uniformly on compact subsets of $\Delta'(z_0, \delta)$. Noting that $1/(f_n - z)$ is a holomorphic function in $\Delta'(z_0, \delta)$ for sufficiently large positive integer n, we can find by (18) and the Maximum Modulus Theorem that there exists some positive constant $M(\delta)$ depending only upon δ such that for sufficiently large positive integer n we have

$$|1/(f_n - z)| \leqslant M(\delta), \tag{19}$$

for all $z \in \bar{\Delta}(z_0, \delta/2)$. From (19) we can find that $\{1/(f_n - z)\}$ and so f_n is normal in $\Delta(z_0, \delta/2)$.

Subcase 1.2 Suppose that $z_0 = 0$ and that there exist a subsequence of $\{f_n\}$, say itself such that for every positive integers $n \geqslant 1$, we have $f_n(z_0) - z_0 = f_n(z_0) \neq 0$. Combining this with (16), we can find that $1/f_n$ is a holomorphic function in $\Delta(z_0, \delta)$ for sufficiently large positive integer n. Therefore, by the Maximum Modulus Theorem we can find that there exists some positive constant $M(\delta)$ depending only upon δ such that for sufficiently large positive integer n we have

$$|1/f_n(z)| \leqslant M(\delta) \tag{20}$$

for all $z \in \bar{\Delta}(z_0, \delta/2)$. From (20) we can see that $\{1/f_n\}$, and so $\{f_n\}$ is normal in $\Delta(z_0, \delta/2)$.

Subcase 1.3 Suppose that there exist a subsequence of $\{f_n\}$, say itself such that for every positive integers $n \geqslant 1$, we have $f_n(z_0) - z_0 = 0$. Then, from the assumption that $R(f_n) - z$ and $f_n - z$ share 0 IM, we can see that

$$R(z_0) = z_0.$$

On the other hand, by the assumptions of Theorem 10 we can see that there exists a value α_2 such that $R(\alpha_2) = \alpha_2$ and $\alpha_2 \neq z_0$. This together with the fact $R(z) \neq z$ for all $z \in \Delta'(z_0, \delta)$ implies that $1/(f_n - \alpha_2)$ is a holomorphic function in $\Delta(z_0, \delta)$ for sufficiently large positive integer n. Moreover, from (16) we have

$$1/(f_n - \alpha_2) \to 0, \tag{21}$$

spherical uniformly on compact subsets of $\Delta'(z_0, \delta)$. From (21) and the Maximum Modulus Theorem we can see that there exists some positive number $M(z_0, \delta)$ depending only upon z_0 and δ such that for sufficiently large positive integer n we have

$$|1/(f_n - \alpha_2)| \leqslant M(z_0, \delta) \tag{22}$$

for all $z \in \bar{\Delta}(z_0, \delta/2)$. From (22) we can see that $\{1/(f_n - \alpha_2)\}$, and so $\{f_n\}$ is normal in $\Delta(z_0, \delta/2)$.

Subcase 2 Suppose that (15) holds and that

$$f(z) = 0$$

for all $z \in \Delta'(z_0, \delta)$. Then by replacing $1/f_n$ with f_n in Subcase 1 and in the same manner as in the proof of Subcase 1 we can prove that $\{f_n\}$ is normal in $\Delta(z_0, \delta/2)$.

Subcase 3 Suppose that (15) holds and that

$$f(z) \not\equiv 0,\infty \tag{23}$$

for all $z \in \Delta'(z_0,\delta)$. First of all, by the principle of isolated zeroes, the assumption that $f - z$ and $R(f) - z$ share 0 IM in $\Delta(z_0,\delta)$ and the fact $R \not\equiv z$ we can find some sufficiently small positive number δ such that $R(z) \neq z$ for all $z \in \Delta'(z_0,\delta)$. we consider the following two subcases:

Subcase 3.1 Suppose that there exist a subsequence of $\{f_n\}$, say itself such that

$$f_n(z_0) \in \{z: R(z) = z\} \tag{24}$$

for all $n \geqslant 1$. Then, there is a subsequence of $\{f_n\}$, say itself such that

$$\lim_{n\to\infty} f_n(z_0) = \hat{z}_j, 1 \leqslant j \leqslant l, \tag{25}$$

where $\hat{z}_j \in \{z: R(z) = z\}$ is some point, say $\hat{z}_j = \hat{z}_1$.

Suppose that $\hat{z}_1 \neq z_0$. Then, from (25) and the fact $R(z) \neq z$ for all $z \in \Delta'(z_0,\delta)$ we can see that for sufficiently large positive integer n, $1/(f_n - z)$ is a holomorphic function in $z \in \Delta(z_0,\delta)$. Next, in the same manner as in Subcase 1.3 we can prove that $\{1/(f_n - z)\}$, and so $\{f_n\}$ is normal in $\Delta(z_0,\delta/2)$.

Suppose that $\hat{z}_1 = z_0$ and that $\{f_n\}$ is not normal at z_0. Then, there exists some neighbourhood of z_0, say $|z - z_0| < \delta$ such that $\{f_n\}$ is not normal in $|z - z_0| < \delta$. Then, by Lemma 1 we can find that there exist points $z_n \to z_0$, $|z_n - z_0| < \delta$, positive numbers ρ_n, $\rho_n \to 0^+$ and a subsequence that belong to $\{f_n\}$, say $\{f_n\}$ itself such that

$$f_n(z_n + \rho_n\zeta) := g_n(\zeta) \to g(\zeta), \tag{26}$$

spherical uniformly on compact subsets of $\mathbf{C}$, where g is some nonconstant meromorphic function of order $\rho(g) \leqslant 2$. Therefore,

$$f_n(z_n + \rho_n\zeta) - (z_n + \rho_n\zeta) = g_n(\zeta) - (z_n + \rho_n\zeta) =: H_n(\zeta) \to g(\zeta) - z_0 \tag{27}$$

and

$$R(f_n(z_n + \rho_n\zeta)) - (z_n + \rho_n\zeta) \to R(g(\zeta)) - z_0, \tag{28}$$

spherical uniformly on compact subsets of $\mathbf{C}$. By the assumption of Theorem 10 we can find that there exist two distinct finite values β_1 and β_2 in the complex plane such that $\beta_1 \neq z_0$ and $\beta_2 \neq z_0$, and such that $R(\beta_j) = \beta_j$ for j = 1,2. Combining this with the assumption that for every $f \in \mathscr{F}$, $f - z$ and $R(f) - z$ share 0 IM in D, we can deduce from (26) ~ (28) and Hurwitz's Theorem that g and $R(g)$ share β_1, β_2 and z_0 IM. Next we prove the following claim:

(i) $g(\zeta) - z_0$ has at most one zero in the complex plane.

We prove the claim (i): Suppose that η_1 and η_2 are two distinct finite values in the complex plane such that $g(\eta_1) = g(\eta_2) = z_0$. Then, for sufficiently small positive number δ satisfying $\delta < |\eta_1 - \eta_2|$ we have $\Delta(\eta_1,\delta/2) \cap \Delta(\eta_2,\delta/2) = \varnothing$, where $\varnothing$ denotes the empty set. Combining this, (26) and Hurwitz's Theorem, we can find that there exist a sequence of points ζ_n such that

$$f_n(z_n + \rho_n\zeta_n) - (z_n + \rho_n\zeta_n) = 0 (\zeta_n \to \eta_1). \tag{29}$$

By the assumption that for every $f \in \mathscr{F}$, $f - z$ and $R(f) - z$ share 0 IM in D, we can get from (29) that

$$R(f_n(z_n + \rho_n\zeta_n)) = R(z_n + \rho_n\zeta_n) = z_n + \rho_n\zeta_n \quad (\zeta_n \to \eta_1). \tag{30}$$

On the other hand, by (24) ~ (25), the suppositions $\hat{z}_1 = z_0$ and $R(z) \neq z$ for all $z \in \Delta'(z_0,\delta)$ we can deduce for sufficiently large positive integer n that

$$z_n + \rho_n\zeta_n = z_0. \tag{31}$$

Similarly, we can find that there exist a sequence of points $\hat{\zeta}_n$ such that for sufficiently large positive integer n we have

$$R(f_n(z_n + \rho_n\hat{\zeta}_n)) = R(z_n + \rho_n\hat{\zeta}_n) = z_n + \rho_n\hat{\zeta}_n \quad (\hat{\zeta}_n \to \eta_2), \tag{32}$$

and

$$z_n + \rho_n\hat{\zeta}_n = z_0. \tag{33}$$

From (31) and (33) we have

$$z_n + \rho_n\zeta_n = z_n + \rho_n\hat{\zeta}_n = z_0,$$

and so

$$\zeta_n = \hat{\zeta}_n = (z_0 - z_n)/\rho_n \tag{34}$$

for sufficiently large positive integer n. From (30) and (32) we have $\zeta_n \to \eta_1$ and $\hat{\zeta}_n \to \eta_2$. This with (34) and the supposition $\eta_1 \neq \eta_2$, we can get a contradiction. This proves the claim (i). Then, by the standard Valiron-Mokhon'ko lemma[21], Nevanlinna's three small functions theorem[4], the assumption deg $R \geqslant 3$ and the fact that g and $R(g)$ share β_1, β_2 and z_0 IM, we have

$$3T(r,g) \leqslant T(r,R(g)) + O(1) \leqslant \overline{N}\left(r,\frac{1}{R(g) - z_0}\right) + \overline{N}\left(r,\frac{1}{R(g) - \beta_1}\right) + \overline{N}\left(r,\frac{1}{R(g) - \beta_2}\right) + O(\log r) =$$

$$\overline{N}\left(r,\frac{1}{g-z_0}\right)+\overline{N}\left(r,\frac{1}{g-\beta_1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_2}\right)+$$

$$O(\log r)\leqslant 2T(r,g)+O(\log r), \qquad (35)$$

and so $T(r,g)=O(\log r)$, which implies that g and $R(g)$ are a nonconstant rational function. Thus (35) can be rewritten as

$$3T(r,g)\leqslant T(r,R(g))+O(1)\leqslant \overline{N}\left(r,\frac{1}{R(g)-z_0}\right)+$$

$$\overline{N}\left(r,\frac{1}{R(g)-\beta_1}\right)+\overline{N}\left(r,\frac{1}{R(g)-\beta_2}\right)+O(1)=$$

$$\overline{N}\left(r,\frac{1}{g-z_0}\right)+\overline{N}\left(r,\frac{1}{g-\beta_1}\right)+\overline{N}\left(r,\frac{1}{g-\beta_2}\right)+O(\log r)\leqslant$$

$$N\left(r,\frac{1}{g-z_0}\right)+N\left(r,\frac{1}{g-\beta_1}\right)+N\left(r,\frac{1}{g-\beta_2}\right)+O(1)\leqslant$$

$$3T(r,g)+O(1),$$

which implies that

$$N_{(2}\left(r,\frac{1}{g-z_0}\right)+N_{(2}\left(r,\frac{1}{g-\beta_1}\right)+N_{(2}\left(r,\frac{1}{g-\beta_2}\right)=$$

$$O(1), \qquad (36)$$

where $N_{(2}(r,1/(g-z_0))$ denotes the counting function of those zeros of $g-z_0$ (counted with proper multiplicities) whose multiplicities are not less than 2, $N_{(2}(r,1/(g-\beta_1))$ and $N_{(2}(r,1/(g-\beta_2))$ have the similar meanings. Therefore, by (36) we can see that g and $R(g)$ share z_0, β_1 and β_2 CM in the complex plane. This together with Lemma 6 gives $g=R(g)$, and so $3T(r,g)\leqslant T(r,R(g))+O(1)\leqslant T(r,g)+O(1)$, i. e., $T(r,g)=O(1)$, which is impossible.

Subcase 3.2 Suppose that there exist a subsequence of $\{f_n\}$, say itself such that

$$f_n(z_0)\notin\{z: R(z)=z\} \qquad (37)$$

for every positive integer $n\geqslant 1$. Then, from (37), the assumption of Theorem 10 and the fact $R(z)\neq z$ for all $z\in\Delta'(z_0,\delta)$ we can deduce $f_n(z)-z\neq 0$ for all $z\in\Delta(z_0,\delta)$, and so $\{1/(f_n-z)\}$ is a family of holomorphic functions in $\Delta(z_0,\delta)$. Moreover, from (15) we have

$$1/(f_n-z)\rightarrow 1/(f-z) \qquad (38)$$

spherical uniformly on compact subsets of $\Delta'(z_0,\delta)$. By (23) we can deduce that $1/(f-z)\not\equiv 0$ in $\Delta'(z_0,\delta)$. This together with (38) and the Hurwitz's Theorem implies that $1/(f-z)$ is holomorphic in $\Delta'(z_0,\delta)$. From (38) and the Maximum Modulus Theorem we can see that there exists some positive constant M that depends only upon f and δ such that for sufficiently large positive integer n, we have

$$|1/(f_n(z)-z)|\leqslant M \qquad (39)$$

for all $\overline{\Delta}(z_0,\delta/2)$. From (39) we can deduce that $\{1/(f_n(z)-z)\}$, and so $\{f_n\}$ is normal in $\Delta(z_0,\delta/2)$.

Theorem 10 is thus completely proved.

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On the λ^*-Logarithmic Type of Analytic Functions Represented by Laplace-Stieltjes Transformation

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Abstract: By using the concept of λ^*-logarithmic type,the analytic function of irregular logarithmic growth which defined by Laplace-Stieltjes transformation converging in right plane is studied. Some relationship on maximum modulus,maximum term,rank of maximum term and the λ^*-logarithmic type are obtained,which extend some results of Dirichlet series.

Key words: Laplace-Stieltjes transform; lower logarithmic type; λ^*-logarithmic type; growth

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涉及复合亚纯函数和不动点的亚纯函数的正规族

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摘要: 假设 f 是一个超越整函数,G 是定义在区域 $D\subset \mathbf{C}$ 上的全纯函数族. 如果对 G 中每个元素 g,$f(g)-\alpha_1$ 在区域 D 上的每个零点重数 $\geqslant 2$,$f(g)-\alpha_2$ 和 $g-\alpha_2$ 在区域 D 上 IM 分担 0,这里 α_1 和 α_2 是 2 个判别的有穷复数,则 G 在区域 D 上是正规的,该结果推广了 Bergweiler 2004 年的一个结果. 同时还证明了:假设 R 是一个次数满足 $\deg R\geqslant 2$($\deg R\geqslant 3$,并且 R 在在复平面上有 3 个判别的有限的不动点)的有理函数,$\mathscr{F}$是一个定义在区域 $D\subset \mathbf{C}$ 上的全纯函数(亚纯函数),并且对 $\mathscr{F}$中每个元素 f,$R\circ f(z)-z$ 和 $f(z)-z$ 在区域 D 上 IM 分担 0,则 $\mathscr{F}$是区域 D 上的正规族,该结果推广了方明亮与袁文俊 2000 年的一个结果,也推广了常建明、方明亮与 L. Zalcman 2005 年的一个结果,并举例说明本文结果从某种意义上来讲是最佳的.

关键词: 亚纯函数; 复合函数; 不动点; 分担值; 正规定则

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