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一类耦合型薛定谔系统径向正解的存在性

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摘要: 利用变分法和椭圆方程理论研究如下的非线性薛定谔方程组:

$$\begin{cases} -\Delta u + u = h(u) + \lambda \frac{2uv^2}{1+u^2v^2}, x \in \mathbf{R}^N, \\ -\Delta v + v = g(v) + \lambda \frac{2u^2v}{1+u^2v^2}, x \in \mathbf{R}^N, \\ u \rightarrow 0, v \rightarrow 0, & |x| \rightarrow +\infty. \end{cases}$$

假设 h 和 g 满足一定的条件, $\exists \lambda_0 \in (0, 1), \forall \lambda \in (0, \lambda_0)$, 得到径向正解的存在性.

关键词: 变分法; 耦合型薛定谔系统; 径向正解

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0 引言和结果

薛定谔方程是量子力学中最重要的方程, 最近十几年来, 对于与物理学密切相关的非线性薛定谔方程解的存在性和解的各种性质的研究, 一直是全世界数学和物理工作者关注的焦点和研究热点.

考虑下面非线性薛定谔方程组:

$$\begin{cases} -\Delta u + u = h(u) + \lambda \frac{2uv^2}{1+u^2v^2}, x \in \mathbf{R}^N, \\ -\Delta v + v = g(v) + \lambda \frac{2u^2v}{1+u^2v^2}, x \in \mathbf{R}^N, \\ u \rightarrow 0, v \rightarrow 0, & |x| \rightarrow +\infty, \end{cases} \quad (1)$$

其中 $N \geq 3, \lambda \geq 0$, 函数 h 和 g 是 Carathéodary 函数. 该系统来源于非线性光学^[1].

近年来, 此类方程组由于在非线性力学和 Bose-Einstein 凝聚等物理中有着非常重要的作用, 引起了广大数学和物理工作者的极大关注. 大量杰出的研究成果不断涌现. 下面简单介绍一下已有的重要结果, 然后给出该文的研究问题.

方程组(1)的解 $(u, v) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ 被称为稳态解; 若 $(u, v) \neq (0, 0)$, 则称之为非平凡稳态解; 若 $u > 0, v > 0$, 则称之为正稳态解; 若 $(u, v) \neq (0, 0)$ 且其能量是极小能量, 则称 (u, v) 是方

程组(1)的基态解; 若基态解 (u, v) 满足 $u > 0, v > 0$, 则称该解为正基态解. 方程组(1)的解对应于 C^2 泛函 $\Phi_\lambda: H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) \rightarrow \mathbf{R}$,

$$\begin{aligned} \Phi_\lambda(u, v) = & \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla u|^2 + |u|^2) dx + \\ & \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla v|^2 + |v|^2) dx - \int_{\mathbf{R}^N} (H(u) + G(v)) dx - \\ & \lambda \int_{\mathbf{R}^N} \ln(1 + u^2v^2) dx \end{aligned}$$

的临界点, 其中 $H(u) := \int_0^u h(s) ds, G(v) := \int_0^v g(s) ds$. 令 $2^* = 2N/(N-2)$, 作如下假设:

- (A₁) $h, g \in C(\mathbf{R}, \mathbf{R}), \lim_{s \rightarrow 0} h(s)/s = \lim_{s \rightarrow 0} g(s)/s = 0$;
- (A₂) $\exists p \in (1, 2^* - 1)$ 满足 $\limsup_{s \rightarrow +\infty} h(s)/s^p < +\infty, \limsup_{s \rightarrow +\infty} g(s)/s^p < +\infty$;
- (A₃) $\sup_{s > 0} 2H(s)/s^2 > 1, \sup_{s > 0} 2G(s)/s^2 > 1$.

称条件(A₁) ~ (A₃)为 Berestycki-Lions 条件^[2], 文献[2]得到方程

$$-\Delta u + u = h(u), \mu \in H^1(\mathbf{R}^N)$$

基态解的存在性. 其它结果参见文献[3-7]. 下面简单回顾几个重要的研究成果. 当 $N \leq 3, h(u) = g(u) = u^3$, 对于充分小的 $\lambda > 0$, A. Ambrosetti 等^[3]得到了系统

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$$\begin{cases} -\Delta u + u = h(u) + \lambda v, & x \in \mathbf{R}^N \\ -\Delta v + v = g(v) + \lambda u, & x \in \mathbf{R}^N \\ u \rightarrow 0, v \rightarrow 0, & |x| \rightarrow +\infty \end{cases} \quad (2)$$

孤立子解的存在性. 在方程(2)中若函数 $h(u) = (1 + a(x)) |u|^{p-1}u$, $g(v) = (1 + b(x)) |v|^{p-1}v$, 则当 $N \geq 1$ 时, 已有解的存在性结果^[8-9]. 当 $\lambda \in [0, 1)$ 时, 文献[10]得到了系统(2)径向解 $u, v \in H^1(\mathbf{R}^N) \cap W_{loc}^{2,p}(\mathbf{R}^N)$ 的存在性. 邹文明等^[11]得到了系统(2)径向正对称正基态解的存在性. 从上面问题回顾中可知, 方程组(2)解的研究已经取得了丰富成果. 本文主要研究一些与方程组(1)密切相关的重要问题, 其中主要应用文献[12]给出的变分方法.

在本文中 C 表示常数(不同地方取值可以不一样), 定义 $L^p(\Omega)$ 的范数为 $\|u\|_p = (\int_{\Omega} |u|^p dx)^{1/p}$, $H^1(\mathbf{R}^N)$ 的内积为 $\langle u, v \rangle := \int_{\mathbf{R}^N} (\nabla u \nabla v + uv) dx$, 范数为 $\|u\|^2 := \langle u, u \rangle$. 记 $H := H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$, 其上的范数为 $\|(u, v)\|_H^2 := \|u\|^2 + \|v\|^2$. 定义 $H_r := H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)$, 其中

$$H_r^1(\mathbf{R}^N) := \{u \in H^1(\mathbf{R}^N) : u \text{ 径向对称}\}.$$

本文主要考虑函数 Φ_λ 限制在空间 H_r 上, 主要结果如下.

定理 1 假设 $N \geq 3$, 函数 h 和 g 满足 $(A_1) \sim (A_3)$, 则 $\exists \lambda_0 \in (0, 1), \forall \lambda \in (0, \lambda_0)$, 方程组(1)存在径向正解 $(u_\lambda, v_\lambda) \in C^2(\mathbf{R}^N) \times C^2(\mathbf{R}^N)$.

1 主要结果的证明

方程组(1)的解对应于 C^2 泛函 $\Phi_\lambda : H_r \rightarrow \mathbf{R}$,

$$\Phi_\lambda(u, v) := \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla u|^2 + |u|^2) dx +$$

$$\frac{1}{2} \int_{\mathbf{R}^N} (|\nabla v|^2 + |v|^2) dx - \int_{\mathbf{R}^N} (H(u) + G(v)) dx - \lambda \int_{\mathbf{R}^N} \ln(1 + u^2 v^2) dx$$

的临界点. 考虑方程组(1)的正解, 不失一般性, 假设 $\forall s \leq 0$,

$$h(s) \equiv g(s) \equiv 0. \quad (3)$$

由 Palais 对称临界点理论可知, Φ_λ 的任意临界点是方程组(1)的径向对称解. 考虑方程

$$-\Delta u + u = h(u), \quad \mu \in H^1(\mathbf{R}^N), \quad (4)$$

$$-\Delta v + v = g(v), \quad \nu \in H^1(\mathbf{R}^N), \quad (5)$$

相应的能量泛函记为

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla u|^2 + |u|^2) dx - \int_{\mathbf{R}^N} H(u) dx,$$

$$u \in H^1(\mathbf{R}^N),$$

$$J(v) = \frac{1}{2} \int_{\mathbf{R}^N} (|\nabla v|^2 + |v|^2) dx - \int_{\mathbf{R}^N} G(v) dx,$$

$$v \in H^1(\mathbf{R}^N).$$

若 h 和 g 满足 $(A_1) \sim (A_3)$, H. Berestycki 等^[2]证明了方程(4) ~ (5) 正基态解的存在性, 且方程的解 u, v 满足 Pohozaev 等式:

$$\int_{\mathbf{R}^N} |\nabla u|^2 dx + 2^* \int_{\mathbf{R}^N} (u^2/2 - H(u)) dx = 0, \quad (6)$$

$$\int_{\mathbf{R}^N} |\nabla v|^2 dx + 2^* \int_{\mathbf{R}^N} (v^2/2 - G(v)) dx = 0. \quad (7)$$

文献[10]证明了方程(4) ~ (5) 的基态解经过平移后是径向的. 由(3)式可知, 方程(4) ~ (5) 的任意非平凡解都是正的. 定义

$$X_1 := \{u \in H_r^1(\mathbf{R}^N) : u \text{ 是方程(4)的正基态解}\},$$

$$X_2 := \{v \in H_r^1(\mathbf{R}^N) : v \text{ 是方程(5)的正基态解}\},$$

$$X := X_1 \times X_2.$$

由文献[12]可知 X_1 和 X_2 在空间 $H_r^1(\mathbf{R}^N)$ 中是紧的. $\forall u_0 \in X_1$ 和 $v_0 \in X_2$, 记方程(4)和(5)的极小能量泛函为 $R_1 := I(u_0)$, $R_2 := J(v_0)$. 不失一般性, 假设 $R_1 \leq R_2$. 由文献[2]可知 $R_1 > 0, R_2 > 0$, 则存在常数 $c_1 > c_2 > 0$ 满足

$$c_2 \leq \|u\|, \|v\| \leq c_1, \forall (u, v) \in X. \quad (8)$$

由文献[13]可得

$$\begin{aligned} R_1 &= \inf_{u \in H^1(\mathbf{R}^N) \setminus \{0\}, I(u)=0} I(u), \\ R_2 &= \inf_{v \in H^1(\mathbf{R}^N) \setminus \{0\}, J(v)=0} J(v), \end{aligned} \quad (9)$$

对于 $t, s > 0$, 定义 $u_0^t := u_0(x/t)$, $v_0^s := v_0(x/s)$, 由(6)和(7)式可知

$$\begin{aligned} I(u_0^t) &= \int_{\mathbf{R}^N} \left(\frac{t^{N-2}}{2} |\nabla u_0|^2 + \frac{t^N}{2} u_0^2 - t^N H(u_0) \right) dx = \\ &= \left(\frac{t^{N-2}}{2} - \frac{(N-2)t^N}{2N} \right) \int_{\mathbf{R}^N} |\nabla u_0|^2 dx. \end{aligned} \quad (10)$$

$$\begin{aligned} J(v_0^s) &= \int_{\mathbf{R}^N} \left(\frac{s^{N-2}}{2} |\nabla v_0|^2 + \frac{s^N}{2} v_0^2 - s^N G(v_0) \right) dx = \\ &= \left(\frac{s^{N-2}}{2} - \frac{(N-2)s^N}{2N} \right) \int_{\mathbf{R}^N} |\nabla v_0|^2 dx. \end{aligned} \quad (11)$$

由于

$$I(u_0^t) = \max_{t>0} I(u_0^t) = I(u_0) = R_1, \quad (12)$$

$$J(v_0^s) = \max_{s>0} J(v_0^s) = J(v_0) = R_2, \quad (13)$$

易证 $\exists 0 < t_0 < 1 < t_1$ 和 $0 < s_0 < 1 < s_1$ 满足

$$I(u_0^t) \leq R_1/2, \forall t \in (0, t_0] \cup [t_1, \infty), \quad (14)$$

$$J(v_0^s) \leq R_1/2, \forall s \in (0, s_0] \cup [s_1, \infty), \quad (15)$$

定义

$$\pi_1(t) := u_0^t, \forall t \in (0, t_1], \pi_1(0) := 0;$$

$$\pi_2(s) := v_0^s, \forall s \in (0, s_1], \pi_2(0) := 0;$$

$$\pi(t, s) := (\pi_1(t), \pi_2(s)).$$

定义 $T := [0, t_1] \times [0, s_1]$ 存在常数 $c > 0$ 满足 $\max_{(t,s) \in T} \|\pi(t, s)\| \leq c$.

定义 $M_\lambda := \inf_{\tilde{\pi} \in \Pi} \max_{(t,s) \in T} \Phi_\lambda(\tilde{\pi}(t, s))$, $N_\lambda := \max_{(t,s) \in T} \Phi_\lambda(\pi(t, s))$ 其中

$$\Pi := \{ \tilde{\pi} \in C(T, H_r) : \max_{(t,s) \in T} \|\tilde{\pi}(t, s)\| \leq 2c_1 + c, \tilde{\pi}(t, s) = \pi(t, s), (t, s) \in T \setminus (t_0, t_1) \times (s_0, s_1) \}.$$

引理 1 $\lim_{\lambda \rightarrow 0} M_\lambda = \lim_{\lambda \rightarrow 0} N_\lambda = M_0 = N_0 = R_1 + R_2$.

证 由 $\lambda \geq 0$ 可知 $\Phi_\lambda(\pi(t, s)) \leq \Phi_0(\pi(t, s))$ 因此

$$N_\lambda \leq N_0 = \max_{(t,s) \in T} \Phi_0(\pi(t, s)) = \max_{t \in [0, t_1]} I(\pi_1(t)) + \max_{s \in [0, s_1]} J(\pi_2(s)) = I(\pi_1(1)) + J(\pi_2(1)) = I(u_0) + J(v_0) = R_1 + R_2.$$

由于 $\pi \in \Pi$ 则 $M_\lambda \leq N_\lambda$ 即

$$\limsup_{\lambda \rightarrow 0} M_\lambda \leq \liminf_{\lambda \rightarrow 0} N_\lambda \leq \limsup_{\lambda \rightarrow 0} N_\lambda \leq N_0.$$

定义

$$\tilde{H}(u) := \begin{cases} \frac{2^* \int_{\mathbb{R}^N} H(u) dx}{\int_{\mathbb{R}^N} (|\nabla u|^2 + 2^* u^2/2) dx} & u \neq 0 \\ 0, & u = 0, \end{cases}$$

$$\tilde{G}(v) := \begin{cases} \frac{2^* \int_{\mathbb{R}^N} G(v) dx}{\int_{\mathbb{R}^N} (|\nabla v|^2 + 2^* v^2/2) dx} & v \neq 0 \\ 0, & v = 0. \end{cases}$$

$\forall \tilde{\pi}(t, s) = (\tilde{\pi}_1(t, s), \tilde{\pi}_2(t, s)) \in \Pi$ 定义

$$\Gamma(\tilde{\pi}) : [t_0, t_1] \times [s_1, s_1] \rightarrow \mathbb{R}^2 \text{ 为}$$

$$\Gamma(\tilde{\pi})(t, s) := (\tilde{H}(\tilde{\pi}_1(t, s)) - \tilde{G}(\tilde{\pi}_2(t, s)),$$

$$\tilde{H}(\tilde{\pi}_1(t, s)) + \tilde{G}(\tilde{\pi}_2(t, s)) - 2).$$

由 (A₁) ~ (A₂) 和 Sobolev 不等式 $|u|_{p+1}^{p+1} \leq C \|u\|^{p+1}$ 易证 \tilde{H}, \tilde{G} 是连续函数. 类似于 (10) 式和 (11) 式易证

$$\Gamma(\pi)(t, s) = \left(\frac{2^* t^2 \int_{\mathbb{R}^N} H(u_0) dx}{\int_{\mathbb{R}^N} (|\nabla u_0|^2 + 2^* t^2 u_0^2/2) dx} - \frac{2^* s^2 \int_{\mathbb{R}^N} G(v_0) dx}{\int_{\mathbb{R}^N} (|\nabla v_0|^2 + 2^* s^2 v_0^2/2) dx} \right) + \frac{2^* t^2 \int_{\mathbb{R}^N} H(u_0) dx}{\int_{\mathbb{R}^N} (|\nabla u_0|^2 + 2^* t^2 u_0^2/2) dx}$$

$$\frac{2^* s^2 \int_{\mathbb{R}^N} G(v_0) dx}{\int_{\mathbb{R}^N} (|\nabla v_0|^2 + 2^* s^2 v_0^2/2) dx} - 2).$$

由 (6) ~ (7) 式可知 $\Gamma(\pi)(1, 1) = (0, 0)$ 则 $\deg(\Gamma(\pi), [t_0, t_1] \times [s_0, s_1], (0, 0)) = 1$. $\forall (t, s) \in \partial [t_0, t_1] \times [s_0, s_1]$, 由 (16) 式可知 $\Gamma(\tilde{\pi})(t, s) = \Gamma(\pi)(t, s) \neq (0, 0)$. 由此可知 $\deg(\Gamma(\tilde{\pi}), [t_0, t_1] \times [s_0, s_1], (0, 0))$ 有定义, 且有

$$\deg(\Gamma(\tilde{\pi}), [t_0, t_1] \times [s_0, s_1], (0, 0)) = \deg(\Gamma(\pi), [t_0, t_1] \times [s_0, s_1], (0, 0)) = 1,$$

则 $\exists (t^*, s^*) \in [t_0, t_1] \times [s_0, s_1]$ 满足 $\Gamma(\tilde{\pi})(t^*, s^*) = (0, 0)$, 即 $\tilde{H}(\tilde{\pi}_1(t^*, s^*)) = \tilde{G}(\tilde{\pi}_2(t^*, s^*)) = 1$. 从而 $\tilde{\pi}_1(t^*, s^*), \tilde{\pi}_2(t^*, s^*)$ 分别满足 (6), (7) 式, 且有 $\tilde{\pi}_1(t^*, s^*) \neq 0, \tilde{\pi}_2(t^*, s^*) \neq 0$ 则由 (9) 式可知

$$\max_{(t,s) \in T} \Phi_0(\tilde{\pi}(t, s)) \geq \Phi_0(\tilde{\pi}(t^*, s^*)) \geq I(\tilde{\pi}_1(t^*, s^*)) + J(\tilde{\pi}_2(t^*, s^*)) \geq R_1 + R_2 = N_0.$$

所以 $M_0 \geq N_0$. 由 (18) 式可知 $M_0 = N_0$.

假设 $\lim_{\lambda \rightarrow 0} M_\lambda < N_0$ 则 $\exists \varepsilon > 0, \lambda_n \rightarrow 0$ 和 $\tilde{\pi}_n = (\tilde{\pi}_{n,1}, \tilde{\pi}_{n,2}) \in \Pi$ 满足

$$\max_{(t,s) \in T} \Phi_{\lambda_n}(\tilde{\pi}_n(t, s)) \leq N_0 - 2\varepsilon.$$

由 Π 定义可知, 存在充分大的自然数 N , 使得

$$\max_{(t,s) \in T} \lambda_n \left| \int_{\mathbb{R}^N} \ln(1 + \tilde{\pi}_{n,1}^2(t, s) \tilde{\pi}_{n,2}^2(t, s)) dx \right| \leq c\lambda_n \leq \varepsilon, \forall n \geq N,$$

则有

$$\max_{(t,s) \in T} \Phi_0(\tilde{\pi}_n(t, s)) \leq \max_{(t,s) \in T} \Phi_{\lambda_n}(\tilde{\pi}_n(t, s)) + \varepsilon \leq N_0 - \varepsilon, \forall n \geq N,$$

与 (19) 式矛盾. 故 $\lim_{\lambda \rightarrow 0} M_\lambda \geq N_0$. 证毕.

由 $X := X_1 + X_2$ 可定义

$$X_d := \{(u, v) \in H_r : \text{dist}((u, v), X) \leq d\},$$

$$\Phi_\lambda^\rho := \{(u, v) \in H_r : \Phi_\lambda(u, v) \leq \rho\}.$$

引理 2 对于充分小的 $d \in (0, \varepsilon_2/2)$, 存在常数 $\delta \in (0, 1)$ 和 $\lambda_0 \in (0, 1)$, $\forall (u, v) \in \Phi_{\lambda_n}^{N_{\lambda_n}} \cap (X_d \setminus X_{d/2})$ 和 $\lambda \in (0, \lambda_0)$ 满足 $\|\Phi'_\lambda(u, v)\| \geq \delta$.

证 假设 $\exists \lambda_n \rightarrow 0$ 和 $(u_n, v_n) \in \Phi_{\lambda_n}^{N_{\lambda_n}} \cap (X_d \setminus X_{d/2})$ 满足 $\|\Phi'_{\lambda_n}(u_n, v_n)\| \rightarrow 0$ 其中 $\Phi_{\lambda_n}^{N_{\lambda_n}} := \{(u, v) \in H_r : \Phi_{\lambda_n}(u, v) \leq N_{\lambda_n}\}$. 由引理 1 可知, $\{(u_n, v_n)\}_{n \geq 1}$ 在空间 H_r 中一致有界. $\forall p \in (2, 2^*)$, 由于 Sobolev 嵌入 $H_r^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ 是紧嵌

入 选取子序列,假定

$$(u_n, v_n) \rightharpoonup (u_0, v_0) \text{ 弱收敛于 } H_r,$$

$(u_n, v_n) \rightarrow (u_0, v_0)$ 强收敛于 $L^p(\mathbf{R}^N)$ $p \in (2, 2^*)$, 则有 $\Phi'_0(u_0, v_0) = 0$ 即 u_0, v_0 分别是方程(4), (5) 的解. 进一步, 由 $(A_1) \sim (A_2)$ 可知

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} h(u_n) u_n dx = \int_{\mathbf{R}^N} h(u_0) u_0 dx.$$

由 $\Phi'_{\lambda_n}(u_n, v_n) \rightarrow 0$ 可知

$$\|u_n\|^2 = \int_{\mathbf{R}^N} h(u_n) u_n dx + o(1) =$$

$$\int_{\mathbf{R}^N} h(u_0) u_0 dx + o(1) = \|u_0\|^2 + o(1),$$

因此 $u_n \rightarrow u_0$ 强收敛于 $H^1_r(\mathbf{R}^N)$ 类似地 $v_n \rightarrow v_0$ 强收敛于 $H^1_r(\mathbf{R}^N)$ 所以 $(u_0, v_0) \in X_d$ 且有 u_0, v_0 不恒为 0 $\mu_0 > 0, \nu_0 > 0$. 由引理 1 可知

$$R_1 + R_2 \leq I(u_0) + J(v_0) = \Phi_0(u_0, v_0) = \lim_{n \rightarrow \infty} \Phi_{\lambda_n}(u_n, v_n) \leq \lim_{n \rightarrow \infty} N_{\lambda_n} = R_1 + R_2,$$

即 $I(u_0) = R_1, J(v_0) = R_2, \mu_0 \in X_1, \nu_0 \in X_2$. 所以 $(u_0, v_0) \in X$ 与 $(u_n, v_n) \in X_{d/2}$ 矛盾. 证毕.

引理 3 存在常数 $\lambda^* \in (0, \lambda_0]$ 和 $\theta > 0, \forall \lambda \in (0, \lambda^*)$ 若 $\Phi_\lambda(\pi(t, s)) \geq M_\lambda - \theta$ 则有 $\pi(t, s) \in X_{d/2}$ 其中 $d \in (0, \min\{c/2, \epsilon_2/2\})$.

证 假设 $\exists \lambda_n \rightarrow 0, \theta_n \rightarrow 0$ 和 $(t_n, s_n) \in T$ 满足 $\Phi_{\lambda_n}(\pi(t_n, s_n)) \geq M_{\lambda_n} - \theta_n, \pi(t_n, s_n) \in X_{d/2}$. (20) 适当选取子序列, 假定 $(t_n, s_n) \rightarrow (t_0, s_0) \in T$ 则由引理 1 可知

$$\Phi_0(\pi(t_0, s_0)) \geq \lim_{n \rightarrow \infty} M_{\lambda_n} = R_1 + R_2. \quad (21)$$

由(19) ~ (21) 和(17) 式可知 $(t_0, s_0) = (1, 1)$. 所以 $\lim_{n \rightarrow \infty} \|\pi(t_n, s_n) - \pi(1, 1)\| = 0$.

然而 $\pi(1, 1) = (u_0, v_0) \in X$ 矛盾. 证毕.

令

$$\theta^* := \min\{\theta/2, R_1/2, \mu\delta^2/8\}. \quad (22)$$

由引理 1 知, $\exists \hat{\lambda} \in (0, \lambda^*]$ 满足

$$|M_\lambda - N_\lambda| < \theta^*, |M_\lambda - (R_1 + R_2)| < \theta^*, \forall \lambda \in (0, \hat{\lambda}). \quad (23)$$

引理 4 $\forall \lambda \in (0, \hat{\lambda}), \exists \{(u_n, v_n)\}_{n \geq 1} \subset \Phi_\lambda^{N_\lambda} \cap X_d$ 满足 $\Phi'_\lambda(u_n, v_n) \rightarrow 0, n \rightarrow \infty$ 在 H_r 中.

证 对于固定的 $\lambda \in (0, \hat{\lambda})$, 假设 $\exists c_\lambda \in (0, 1)$ 满足 $\|\Phi'_\lambda(u, v)\| \geq c_\lambda$ 在 $\Phi_\lambda^{N_\lambda} \cap X_d$ 中, 则存在定义在 $\Phi_\lambda^{N_\lambda} \cap X_d$ 的某邻域 U_λ 上的伪梯度向量场 $V_\lambda, \forall (u, v) \in U_\lambda$ 满足

$$\|V_\lambda(u, v)\| \leq 2 \min\{1, \|\Phi'_\lambda(u, v)\|\}, \langle \Phi'_\lambda(u, v), V_\lambda(u, v) \rangle \geq \min\{1, \|\Phi'_\lambda(u, v)\|\}$$

$$v) \| \|\Phi'_\lambda(u, v)\|.$$

令 φ_λ 是定义在 H_r 上的 Lipschitz 连续函数, 满足 $\varphi_\lambda \in [0, 1]$ 在 $H_r \setminus U_\lambda$ 上 $\varphi_\lambda \equiv 0$; 在 $\Phi_\lambda^{N_\lambda} \cap X_d$ 上 $\varphi_\lambda \equiv 1$. 令 ψ_λ 是定义在 \mathbf{R} 上的 Lipschitz 连续函数, 满足 $\psi_\lambda \in [0, 1]$ 在 $|x - M_\lambda| \geq \theta$ 上 $\varphi_\lambda(x) \equiv 0$; 在 $|x - M_\lambda| \leq \theta/2$ 上 $\varphi_\lambda(x) \equiv 1$. 令

$$f_\lambda(u, v) := \begin{cases} -\varphi_\lambda(u, v)\psi_\lambda(\Phi_\lambda(u, v))V_\lambda(u, v), & (u, v) \in U_\lambda, \\ 0, & (u, v) \in H_r \setminus U_\lambda, \end{cases}$$

则对如下初值问题存在全局解 $\Psi_\lambda: H_r \times [0, \infty) \rightarrow H_r$,

$$\begin{cases} \frac{d}{dy} \Psi_\lambda(u, v, y) = f_\lambda(\Psi_\lambda(u, v, y)), \\ \Psi_\lambda(u, v, 0) = (u, v). \end{cases}$$

易证 Ψ_λ 满足如下性质:

- (i) 若 $y = 0$ 或者 $(u, v) \in H_r \setminus U_\lambda$ 或者 $|\Phi_\lambda(u, v) - M_\lambda| \geq \theta$ 则有 $\Psi_\lambda(u, v, y) = (u, v)$;
- (ii) $\|\frac{d}{dy} \Psi_\lambda(u, v, y)\| \leq 2$;
- (iii) $\frac{d}{dy} \Phi_\lambda(\Psi_\lambda(u, v, y)) = \langle \Phi'_\lambda(\Psi_\lambda(u, v, y)), f_\lambda(\Psi_\lambda(u, v, y)) \rangle \leq 0$.

下证 $\forall (t, s) \in T, \exists y_{t,s} \in [0, \infty)$ 满足 $\Phi_\lambda(\pi(t, s), y_{t,s}) \in \Phi_\lambda^{M_\lambda - \theta^*}$.

假设 $\exists (t, s) \in T, \forall y \geq 0$ 满足 $\Phi_\lambda(\Phi_\lambda(\pi(t, s), y)) > M_\lambda - \theta^*$. 由于 $\theta^* < \theta$ 则由引理 3 可知 $\pi(t, s) \in X_{d/2}$. 由于 $\Phi_\lambda(\pi(t, s)) \leq N_\lambda < M_\lambda + \theta^*$, 由性质 (iii) 可知

$$M_\lambda - \theta^* < \Phi_\lambda(\Psi_\lambda(\pi(t, s), y_{t,s})) \leq N_\lambda < M_\lambda + \theta^*, \forall y \geq 0.$$

由此可见 $\psi_\lambda(\Phi_\lambda(\Psi_\lambda(\pi(t, s), y))) \equiv 1$. 若 $\forall y > 0, \Psi_\lambda(\pi(t, s), y) \in X_d$, 则有 $\varphi_\lambda(\Psi_\lambda(\pi(t, s), y)) \equiv 1$ 且 $\forall y > 0, \|\Phi'_\lambda(\Psi_\lambda(\pi(t, s), y))\| \geq c_\lambda$ 则

$$\Phi_\lambda(\Psi_\lambda(\pi(t, s), \frac{\theta}{c_\lambda})) \leq M_\lambda + \frac{\theta}{2} - \int_0^{\theta/c_\lambda} c_\lambda^2 dt \leq M_\lambda - \frac{\theta}{2},$$

矛盾. 所以 $\exists y_{t,s} \in [0, \infty)$ 满足 $\Phi_\lambda(\pi(t, s), y_{t,s}) \notin X_d$. 由于 $\pi(t, s) \in X_{d/2}$, 则 $\exists 0 < y_{t,s}^1 < y_{t,s}^2 \leq y_{t,s}$, 使得 $\Phi_\lambda(\pi(t, s), y_{t,s}^1) \in \partial X_{d/2}, \Phi_\lambda(\pi(t, s), y_{t,s}^2) \in \partial X_d, \forall y \in (y_{t,s}^1, y_{t,s}^2), \Phi_\lambda(\pi(t, s), y) \in X_d \setminus X_{d/2}$. 由引理 2 可知, $\forall y \in (y_{t,s}^1, y_{t,s}^2), \|\Phi'_\lambda(\Psi_\lambda(\pi(t, s), y))\| \geq \delta$ 则由性质 (ii) 可知

$$\epsilon/2 \leq \|\Phi_\lambda(\pi(t, s), y_{t,s}^1) - \Phi_\lambda(\pi(t, s), y_{t,s}^2)\| \leq 2|y_{t,s}^1 - y_{t,s}^2|,$$

即有 $y_{t,s}^2 - y_{t,s}^1 \geq \epsilon/4$. 由此可见

$$\Phi_\lambda(\Psi_\lambda(\pi(t, s), y_{t,s}^2)) \leq \Phi_\lambda(\Psi_\lambda(\pi(t, s), y_{t,s}^1)) +$$

$\int_{y_1^1}^{y_2^1} \frac{d}{dy} \Phi_\lambda(\Psi_\lambda(u, v, y)) dy \leq M_\lambda + \theta^* - \delta^2(y_{i,s}^2 - y_{i,s}^1) \leq M_\lambda + \theta^* - \varepsilon \delta^2/4 \leq M_\lambda - \theta^*$,
 矛盾. 则由上述分析可定义 $Q(t, s) := \inf\{y \geq 0: \Phi_\lambda(\Psi_\lambda(\pi(t, s), y)) \leq M_\lambda - \theta^*\}$. 令 $\tilde{\pi}(t, s) := \Psi_\lambda(\pi(t, s), Q(t, s))$, 则 $\forall (t, s) \in T$, 有 $\Phi_\lambda(\tilde{\pi}(t, s)) \leq M_\lambda - \theta^*$.

下证 $\tilde{\pi}(t, s) \in \Pi$.

$\forall (t, s) \in T \setminus (t_0, t_1) \times (s_0, s_1)$, 由 $R_1 \leq R_2$, (12) ~ (15), (22) ~ (23) 式可知

$\Phi_\lambda(\pi(t, s)) \leq \Phi_0(\pi(t, s)) = I(\pi_1(t)) + J(\pi_2(s)) \leq R_1/4 + R_2 \leq R_1 + R_2 - 3\theta^* < M_\lambda - \theta^*$,
 则有 $Q(t, s) = 0$, 所以 $\tilde{\pi}(t, s) = \pi(t, s)$.

由(11) 式可知, 只需证 $\forall (t, s) \in T, \|\tilde{\pi}(t, s)\| \leq 2c_1 + c$, 且有 $Q(t, s)$ 关于 (t, s) 是连续的. $\forall (t, s) \in T$ 若 $\Phi_\lambda(\pi(t, s)) \leq M_\lambda - \theta^*$ 则有 $Q(t, s) = 0$ 故 $\tilde{\pi}(t, s) = \pi(t, s)$ 进一步, 由(19) 式可知 $\|\tilde{\pi}(t, s)\| \leq c < 2c_1 + c$.

若 $\Phi_\lambda(\pi(t, s)) > M_\lambda - \theta^*$ 则 $\pi(t, s) \in X_{d/2}$ 和 $M_\lambda - \theta^* < \Phi_\lambda(\Psi_\lambda(\pi(t, s), y)) \leq$

$N_\lambda < M_\lambda + \theta^*, \forall y \in [0, Q(t, s))$,

则 $\forall y \in [0, Q(t, s))$, 有 $\psi_\lambda(\Phi_\lambda(\Psi_\lambda(\pi(t, s), y))) \equiv 1$. 若 $\Psi_\lambda(\pi(t, s), Q(t, s)) \notin X_d$ 则存在上述 $0 < y_{i,s}^1 < y_{i,s}^2 < Q(t, s)$. 由上述证明可知 $\Phi_\lambda(\Psi_\lambda(\pi(t, s), y_{i,s}^1)) \leq M_\lambda - \theta^*$ 与 $Q(t, s)$ 的定义矛盾. 所以 $\tilde{\pi}(t, s) = \Psi_\lambda(\pi(t, s), Q(t, s)) \in X_d$. 故 $\exists (u, v) \in X$ 满足 $\|\tilde{\pi}(t, s) - (u, v)\| \leq \varepsilon \leq c/2$. 由(8) 式可知

$$\|\tilde{\pi}(t, s)\| \leq \|(u, v)\| + c/2 < 2c_1 + c.$$

下证 $Q(t, s)$ 关于 (t, s) 是连续的. 选取 $\forall (t_0, s_0) \in T, Q(t_0, s_0) = 0$, 假定 $\Phi_\lambda(\tilde{\pi}(t, s)) < M_\lambda - \theta^*$ 则有 $\Phi_\lambda(\pi(t, s)) < M_\lambda - \theta^*$. 由 π 的连续性可知, $\exists a > 0, \forall (t, s) \in (t_0 - a, t_0 + a) \times (s_0 - a, s_0 + a) \cap T$ 则有 $\Phi_\lambda(\tilde{\pi}(t, s)) < M_\lambda - \theta^*$, 即 $Q(t, s) = 0$ 和 Q 在 (t_0, s_0) 是连续的. 假设 $\Phi_\lambda(\tilde{\pi}(t_0, s_0)) = M_\lambda - \theta^*$ 则由以上证明可知 $\tilde{\pi}(t_0, s_0) = \Psi_\lambda(\pi(t_0, s_0), Q(t_0, s_0)) \in X_d$, 所以

$\|\Phi'_\lambda(\Psi_\lambda(\pi(t_0, s_0), Q(t_0, s_0)))\| \geq c_\lambda > 0$, 则 $\forall b > 0$, 有 $\Phi_\lambda(\Psi_\lambda(\pi(t_0, s_0), Q(t_0, s_0) + b)) < M_\lambda - \theta^*$. 由 Ψ_λ 的连续性可知, $\exists a > 0, \forall (t, s) \in (t_0 - a, t_0 + a) \times (s_0 - a, s_0 + a) \cap T$, 则有 $\Phi_\lambda(\Psi_\lambda(\pi(t, s), Q(t_0, s_0) + a)) < M_\lambda - \theta^*$, 所以

$Q(t, s) \leq Q(t_0, s_0) + b$. 因此,
 $0 \leq \limsup_{(t,s) \rightarrow (t_0,s_0)} Q(t, s) \leq Q(t_0, s_0)$.

若 $Q(t_0, s_0) = 0$ 则有
 $\limsup_{(t,s) \rightarrow (t_0,s_0)} Q(t, s) = Q(t_0, s_0)$.

若 $Q(t_0, s_0) > 0$ 则 $\forall b \in (0, Q(t_0, s_0))$, 同理可得 $\Phi_\lambda(\Psi_\lambda(\pi(t_0, s_0), Q(t_0, s_0) - b)) > M_\lambda - \theta^*$, 由 Ψ_λ 的连续性可知

$$\liminf_{(t,s) \rightarrow (t_0,s_0)} Q(t, s) \geq Q(t_0, s_0).$$

故 Q 在 (t_0, s_0) 是连续的.

由上证明过程可知, $\tilde{\pi}(t, s) \in \Pi$, 且有 $\max_{(t,s) \in T} \Phi_\lambda(\tilde{\pi}(t, s)) \leq M_\lambda - \theta^*$, 则与 M_λ 的定义矛盾. 证毕.

定理 1 的证明 $\forall \lambda \in (0, \hat{\lambda})$, 由引理 4 和(8) 式可知, $\exists \{(u_n, v_n)\}_{n \geq 1} \subset \Phi_\lambda^{-1} \cap X_d$ 满足

$\Phi'_\lambda(u_n, v_n) \rightarrow 0, n \rightarrow \infty$ 在 H_r 中,
 且有 $\{(u_n, v_n)\}_{n \geq 1}$ 在 H_r 中一致有界. 适当选取子序列可得

$(u_n, v_n) \rightarrow (u_0, v_0)$ 弱收敛于 H_r 中, $(u_n, v_n) \rightarrow (u_0, v_0)$ 强收敛于 $L^p(\mathbf{R}^N)$ 中 $p \in (2, 2^*)$,
 则有 $\Phi'_\lambda(u_0, v_0)$ 和 (u_0, v_0) 是方程组(1) 的一个解. 另一方面, 由条件 $(A_1) \sim (A_2)$ 可得

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} (h(u_n)u_n + g(v_n)v_n) dx =$$

$$\int_{\mathbf{R}^N} (h(u_0)u_0 + g(v_0)v_0) dx.$$

由 $\Phi'_\lambda(u_n, v_n)(u_n, v_n) = o(1)$ 可得

$$\int_{\mathbf{R}^N} (|\nabla u_n|^2 + |u_n|^2 + |\nabla v_n|^2 + |v_n|^2 - 2\lambda \ln(1 + u_n^2 v_n^2)) dx = \int_{\mathbf{R}^N} (h(u_n)u_n + g(v_n)v_n) dx + o(1) =$$

$$\int_{\mathbf{R}^N} (h(u_0)u_0 + g(v_0)v_0) dx + o(1) = \int_{\mathbf{R}^N} (|\nabla u_0|^2 + |u_0|^2 + |\nabla v_0|^2 + |v_0|^2 - 2\lambda \ln(1 + u_0^2 v_0^2)) dx,$$

则 $(u_n, v_n) \rightarrow (u_0, v_0)$ 强收敛于 H_r 中, 故

$$(u_0, v_0) \in X_d.$$

下证 $u_0, v_0 > 0$. 事实上, 由方程组(1) 可知 $u_0 \neq 0, v_0 \neq 0$. 假设 $\{x \in \mathbf{R}^N: u_0(x) < 0\} \neq \emptyset$. 由标准正则性理论知, $\forall p < +\infty, u_0, v_0 \in W_{loc}^{2,p}(\mathbf{R}^N)$ 则

$$-\Delta u_0 + u_0 = h(u_0) + \lambda \frac{2u_0 v_0^2}{1 + u_0^2 v_0^2}, \text{ a. e. } x \in \mathbf{R}^N,$$

且有 $u_0, v_0 \in C_{loc}^{1,\alpha}(\mathbf{R}^N), \alpha \in (0, 1)$. 故 $\Delta u_0 \in C(\mathbf{R}^N)$.

令 $u_0(x_0) = \inf_{x \in \mathbf{R}^N} u_0(x) < 0$. 下证 $\Delta u_0(x_0) \geq 0$.

事实上, 若 $\Delta u_0(x_0) < 0$ 则 $\exists r > 0$ 满足

$$u_0(x) < 0, \Delta u_0(x) < 0, \forall x \in B(x_0, r),$$

由最大值原则^[14] 可知 $\inf_{B(x_0, r)} u_0(x) \geq \inf_{\partial B(x_0, r)} u_0(x)$. 由

$u_0(x_0) = \inf_{x \in \mathbf{R}^N} u_0(x)$ 和强最大值原则^[14] 可得 μ_0 在 $B(x_0, r)$ 中恒为常数, 所以 $\Delta u_0(x_0) = 0$, 矛盾. 由此可知 $\Delta u_0(x_0) \geq 0$. 若 $h(u_0(x_0)) = 0$ 则

$$-\Delta u_0(x_0) + u_0(x_0) = \lambda \frac{2u_0(x_0)v_0^2(x_0)}{1 + u_0^2(x_0)v_0^2(x_0)} \leq u_0(x_0) < \lambda u_0(x_0),$$

即 $v_0(x_0) < u_0(x_0) = \inf_{x \in \mathbf{R}^N} u_0(x)$ 或者 $v_0(x_0) > u_0(x_0) = \inf_{x \in \mathbf{R}^N} u_0(x)$, 则得到 $\inf_{x \in \mathbf{R}^N} v_0(x) < \inf_{x \in \mathbf{R}^N} u_0(x)$ 或 $\inf_{x \in \mathbf{R}^N} v_0(x) > \inf_{x \in \mathbf{R}^N} u_0(x)$; 同理可得 $\inf_{x \in \mathbf{R}^N} v_0(x) > \inf_{x \in \mathbf{R}^N} u_0(x)$, 或者 $\inf_{x \in \mathbf{R}^N} v_0(x) < \inf_{x \in \mathbf{R}^N} u_0(x)$, 矛盾. 所以 $\{x \in \mathbf{R}^N : u_0(x) < 0\} = \emptyset$, $\mu_0 \geq 0$. 同理 $\nu_0 \geq 0$. 由强最大值原则可得 $u_0, \nu_0 > 0$.

因此, (u_0, ν_0) 是方程组(1) 的一个径向对称正解. 由标准正则估计可得 $u_0, \nu_0 \in C^2(\mathbf{R}^N)$. 进一步, $\Phi_\lambda(u_0, \nu_0) \leq N_\lambda$.

令 $\lambda_n \in (0, \lambda)$ 满足 $\lambda_n \rightarrow 0$. 重复上述分析可得 $(u_{\lambda_n}, \nu_{\lambda_n}) \rightarrow (u^*, \nu^*)$ 强收敛于 H_r 中, 其中 $u^* \in X_1, \nu^* \in X_2$, 即 u^*, ν^* 分别为方程(4) 和(5) 径向基态正解. 证毕.

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The Existence of Positive Radial Solutions for a Class of Coupled Schrödinger System

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Abstract: The following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + u = h(u) + \lambda \frac{2uv^2}{1 + u^2v^2}, & x \in \mathbf{R}^N \\ -\Delta v + v = g(v) + \lambda \frac{2u^2v}{1 + u^2v^2}, & x \in \mathbf{R}^N \\ u \rightarrow 0, \nu \rightarrow 0, & |x| \rightarrow +\infty \end{cases}$$

is studied. Under some assumptions on h and g , there is $\lambda_0 \in (0, 1)$, such that for any $\lambda \in (0, \lambda_0)$, the existence of positive radial solutions is obtained.

Key words: variational methods; coupled Schrödinger system; positive radial solutions (责任编辑:王金莲)