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The Entire Solutions of Homogeneous Linear Partial Differential Equations of the Second Order Related to Products of Special Functions

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Abstract: By using value distribution theory in heigh dimensional spaces, special functions and classic ordinary differiantial equations, some homogenius linear partial differiantial equations of second order are studied by characterizing entire solutions related closely products of special functions, and a new direction of partial differential equations is enhibited.

Key words: homogeneous linear partial differential equations of the second order; special function; entire solutions

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0 Introduction and Main Results

Hilbert´s 19th problem conjectures that if all coefficients $a_k = a_k(t,z)$ of the following homogeneous linear partial differential equations of the second order in two independent real variables t and z

$$a_0\frac{\partial^2 u}{\partial t^2}+2a_1\frac{\partial^2 u}{\partial t\partial z}+a_2\frac{\partial^2 u}{\partial z^2}+a_3\frac{\partial u}{\partial t}+a_4\frac{\partial u}{\partial z}+a_6 u=0 \quad (1)$$

are analytic on t and z, then any solution $u=u(t,z)$ of an elliptic equation of the form (1) also is analytic on its existing region, which was confirmed by S. N. Bernšteǐn[1]. After being influenced by Cauchy-Kowalewski´s existence theorem, H. Lewy[2] gave a simple proof by extending t and z to a domain of $\mathbf{C}^2$.

By substituting polar coordinates

$$\rho=x^2+y^2,\tau=(x^2-y^2)/(x^2+y^2)$$

in the partial differential equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{4\mu+1}{x}\frac{\partial U}{\partial x}+\frac{4v+1}{y}\frac{\partial U}{\partial y}+k[4\lambda-k(x^2+y^2)]U=0, \quad (2)$$

P. Henrici[3] changed the equation (2) into the form

$$\rho^2\frac{\partial^2 v}{\partial\rho^2}+(1-\tau^2)\frac{\partial^2 v}{\partial\tau^2}+2(\mu+v+1)\rho\frac{\partial v}{\partial\rho}+2[\mu-v-(\mu+v+1)\tau]\frac{\partial v}{\partial\tau}+k\rho(\lambda-\frac{k\rho}{4})v=0$$

and then by the usual separation method $v(\rho,\tau)=R(\rho)T(\tau)$, found that $R(\rho)$ and $T(\tau)$ have to satisfy separately the equations

$$\frac{d^2R}{d\rho^2}+2(\mu+v+1)\frac{1}{\rho}\frac{dR}{d\rho}+(-\frac{s}{\rho^2}+\frac{k\lambda}{\rho}-\frac{k^2}{4})R=0 \quad (3)$$

and

$$(1-\tau^2)\frac{d^2T}{d\tau^2}+2[\mu-v-(\mu+v+1)\tau]\frac{dT}{d\tau}+sT=0, \quad (4)$$

where s is a separation parameter. Writing $s=n(2\mu+2v+n+1)$, he found that solutions of (3) which are regular near $\rho=0$ are represented for $n=0,1,2,\cdots$ by

$$R(\rho)=\rho^{-\mu-v-1}M_{\lambda,\mu+v+1/2+n}(k\rho)$$

where M denotes the Whittaker function of the first kind, while (4) has for the same values of s the Jacobi polynomial solution $T(\tau)=P_n^{(2v,2\mu)}(\tau)$ in the notation of G. Szegö[4], and further studied some analytic prop-

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erties of the functions

$$u(t,z)=R(zt)\,T((z^2+t^2)/(2zt))$$

considered as functions of the two complex variables $z=x+\mathrm{i}y$ and $t=x-\mathrm{i}y$, which satisfy the differential equation

$$\frac{\partial^2 u}{\partial t\partial z}+\frac{2\mu+1/2}{z+t}\left\{\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\right\}-\frac{2v+1/2}{z-t}\left\{\frac{\partial u}{\partial z}-\frac{\partial u}{\partial t}\right\}+k\left(\lambda-\frac{kzt}{4}\right)u=0.$$

Several authors[5-8] studied basic properties and characterization of meromorphic solutions on first-order partial differential equations (or system). Recently, we[9-13] study entire (or meromorphic) solutions of homogeneous linear partial differential equations (1) of the second order in two independent complex variables t and z, where $a_k=a_k(t,z)$ are holomorphic functions for $(t,z)\in\Sigma$, in which Σ is a region on $\mathbf{C}^2$.

Here we will characterize analytic solutions for a series of special cases of (1). For simple, we focus on entire solutions of these kind of equations.

1 Series Expansions Involving Jacobi Polynomials

First of all, we make some remarks for Jacobi polynomials. For complex numbers $a,b,c\in\mathbf{C}$, the hypergeometric function is defined by the Gauss series (or hypergeometric series)

$$F(a;b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n,$$

which is convergent when $|z|<1$, and satisfies Gaussian differential equation

$$z(1-z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\{c-(a+b+1)z\}\frac{\mathrm{d}w}{\mathrm{d}z}-abw=0\qquad(5)$$

where $(\alpha)_n$ is Pochhammer´s symbol

$$(\alpha)_n=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & n\geqslant1,\\ 1, & n=0.\end{cases}$$

The equation (5) also has the solution

$$z^{1-c}F(a+1-c,b+1-c;2-c;z).$$

By using the transformation $z=(1-t)/2$ and setting $a=-n$, $b=\alpha+\beta+n+1$, $c=\alpha+1$, one transfers the (5) into the form

$$(1-t^2)\,\mathrm{d}^2v/\mathrm{d}t^2-\{\alpha-\beta+(\alpha+\beta+2)t\}\,\mathrm{d}v/\mathrm{d}t+n(\alpha+\beta+n+1)v=0$$

with solutions

$$P_n^{(\alpha,\beta)}(t)=(\alpha+1)_nF(-n,\alpha+\beta+n+1;\alpha+1;(1-t)/2)/n!,$$

$$Q_n^{(\alpha,\beta)}(t)=((1-t)/2)^{-\alpha}F(-n-\alpha,\beta+n+1;1-\alpha;(1-t)/2).$$

G. Darboux[14] obtained an asymptotic formula

$$P_n^{(\alpha,\beta)}(t)=P^{(\alpha,\beta)}(t)\,n^{-1/2}\omega^n(t)\{1+p_n^{(\alpha,\beta)}(t)\}\qquad(6)$$

for $n\geqslant1$, where $\omega(t)$ is the inverse of Zukowski transformation $t=(\omega+\omega^{-1})/2$ for which $\omega(\infty)=\infty$, $P^{(\alpha,\beta)}(t)\neq0$ and $\{p_n^{(\alpha,\beta)}(t)\}_{n=1}^{\infty}$ are analytic functions holomorphic in the region $\mathbf{C}-[-1,1]$ and such that $\lim\limits_{n\to\infty}p_n^{(\alpha,\beta)}(t)=0$ uniformly on every compact subset of this region.

For the case $\alpha,\beta>-1$, the corresponding functions of second kind $\{Q_n^{(\alpha,\beta)}(t)\}_{n=1}^{\infty}$ can be defined in the region $\mathbf{C}-[-1,1]$ by the equalities

$$Q_n^{(\alpha,\beta)}(t)=-\int_{-1}^{1}\frac{(1-\zeta)^{\alpha}(1+\zeta)^{\beta}P_n^{(\alpha,\beta)}(\zeta)}{\zeta-t}\mathrm{d}\zeta,$$

$n=0,1,2,\cdots$

which satisfy the following asymptotic formula:

$$Q_n^{(\alpha,\beta)}(t)=Q^{(\alpha,\beta)}(t)\,n^{-1/2}\omega^{-n-1}(t)\{1+q_n^{(\alpha,\beta)}(t)\}\qquad(7)$$

for $n\geqslant1$, where $Q^{(\alpha,\beta)}(t)\neq0$ and $\{q_n^{(\alpha,\beta)}(t)\}_{n=1}^{\infty}$ are holomorphic functions in the region $\mathbf{C}-[-1,1]$ such that $\lim\limits_{n\to\infty}q_n^{(\alpha,\beta)}(t)=0$ uniformly on every compact subset of this region[4].

Further, if $1<r<\infty$, we denote by $E(r)$ the interior of the ellipse $\gamma(r)=\{t\in\mathbf{C}\,|\,|\omega(t)|=r\}$ and we assume by definition that $E(\infty)=\mathbf{C}$. Basing on Christoffel-Darboux formula for Jacobi polynomials and functions of second kind[4], and on the asymptotic formulas (6) and (7), a trivial application of the Cauchy integral formula leads to the following result[15-18]:

Theorem 1 Let $\alpha,\beta>-1$, $1<R\leqslant\infty$ and let f be a holomorphic function in the region $E(R)$. Then f can be represented in this region by a series

$$f(z)=\sum_{n=0}^{\infty}a_nP_n^{(\alpha,\beta)}(z)$$

with the coefficients

$$a_n=\frac{1}{2\pi\mathrm{i}I^{(\alpha,\beta)}}\int_{\gamma(r)}f(\zeta)\,Q_n^{(\alpha,\beta)}(\zeta)\,\mathrm{d}\zeta,\quad n=0,1,2,\cdots$$

for $1<r<R$, where

$$I^{(\alpha,\beta)}=\begin{cases}\dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, & n\geqslant1,\\[2ex] \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & n=0.\end{cases}$$

Moreover, P. Rusev[17] remarked that Theorem 1 is valid also for Jacobi polynomials of arbitrary complex numbers α,β such that $\alpha,\beta,\alpha+\beta\neq -1,-2,\cdots$[19-20]. In the sequel, we will use Jacobi polynomials of this kind of parameters α,β.

1.1 Products of Jacobi Polynomials

Here we consider the following partial differential equation

$$(1-t^2)\,\partial^2u/\partial t^2-(1-z^2)\,\partial^2u/\partial z^2-\{\alpha-\beta+(\alpha+\beta+2)t\}\,\partial u/\partial t+\{\mu-v+(\mu+v+2)z\}\,\partial u/\partial z=0 \quad (8)$$

with $\mu+v=\alpha+\beta$, where α,β (resp. μ,v) satisfy expansion conditions of analytic functions (see Theorem 1 and the remark after it).

By using the usual separation method $u(t,z)=v(t)w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the equations

$$(1-t^2)\frac{d^2v}{dt^2}-[\alpha-\beta+(\alpha+\beta+2)t]\frac{dv}{dt}+sv=0 \quad (9)$$

and

$$(1-z^2)\frac{d^2w}{dz^2}-[\mu-v+(\mu+v+2)z]\frac{dw}{dz}+sw=0 \quad (10)$$

where s is a separation parameter. Writing $s=n(\alpha+\beta+n+1)$, the solutions of (9) which are regular near $t=0$ are represented for $n=0,1,2,\cdots$ by $v(t)=P_n^{(\alpha,\beta)}(t)$, while (10) has for the same values of s the Jacobi polynomial solution $w(z)=P_n^{(\mu,v)}(z)$, and hence (8) have polynomial solutions $u(t,z)=P_n^{(\alpha,\beta)}(t)P_n^{(\mu,v)}(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result characterizes all entire solutions of (8).

Theorem 2 The partial differential equation (8) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}c_nP_n^{(\alpha,\beta)}(t)P_n^{(\mu,v)}(z) \quad (11)$$

such that $\limsup\limits_{n\to\infty}|c_n|^{1/n}=0$.

Proof Let $f(t,z)$ be an entire solution of (8). For any $t\in\mathbf{C}$, we obtain an entire function $f_t(z)=f(t,z)$ on $z\in\mathbf{C}$ which has a Neumann expansion[15-18]

$$f_t(z)=\sum_{n=0}^{\infty}v_n(t)P_n^{(\mu,v)}(z)$$

such that

$$\limsup_{n\to\infty}|v_n(t)|^{1/n}=0. \quad (12)$$

Set $p_{\alpha,\beta}(t)=\alpha-\beta+(\alpha+\beta+2)t$, $s=n(\alpha+\beta+n+1)$. Since $f(t,z)$ is a solution of (8), we find

$$0=(1-t^2)\frac{\partial^2 f}{\partial t^2}-(1-z^2)\frac{\partial^2 f}{\partial z^2}-p_{\alpha,\beta}(t)\frac{\partial f}{\partial t}+p_{\mu,v}(z)\frac{\partial f}{\partial z}=\sum_{n=0}^{\infty}\Big\{\Big((1-t^2)\frac{d^2v_n(t)}{dt^2}-p_{\alpha,\beta}(t)\frac{dv_n(t)}{dt}+sv_n(t)\Big)P_n^{(\mu,v)}(z)-v_n(t)\Big((1-z^2)\frac{d^2P_n^{(\mu,v)}(z)}{dz^2}-p_{\mu,v}(z)\frac{dP_n^{(\mu,v)}(z)}{dz}+sP_n^{(\mu,v)}(z)\Big)\Big\},$$

which means

$$(1-t^2)d^2v_n(t)/dt^2-p_{\alpha,\beta}(t)dv_n(t)/dt+sv_n(t)=0.$$

Therefore, there exists constants c_n,d_n such that $v_n(t)=c_nP_n^{(\alpha,\beta)}(t)+d_nQ_n^{(\alpha,\beta)}(t)$. Note that $v_n(t)$ are entire functions, so that $d_n=0$, and hence

$$f_t(z)=\sum_{n=0}^{\infty}c_nP_n^{(\alpha,\beta)}(t)P_n^{(\mu,v)}(z).$$

By using asymptotic formula of $P_n^{(\alpha,\beta)}(t)$, we easily find

$$\lim_{n\to\infty}|P_n^{(\alpha,\beta)}(t)|^{1/n}=\left|t+\sqrt{t^2+1}\right|,$$

and hence the condition (12) may be replaced by

$$\limsup_{n\to\infty}|c_n|^{1/n}=0.$$

Conversely, it is easy to check that the entire function defined by (11) satisfies (8).

Remark 1 The partial differential equation (8) have meromorphic solutions. For example, let $P_n(t)=P_n^{(0,0)}(t)$ be the Legendre´s polynomial of degree n and let $Q_n(t)$ be the Legendre´s function of second kind of degree n. Then

$$1/(z-t)=\sum_{n=0}^{\infty}(2n+1)Q_n(z)P_n(t)$$

is a solution of (8) with $\alpha=\beta=\mu=v=0$.

1.2 Products of Legendre´s Polynomials and Bessel Polynomials

We consider the following partial differential equation

$$t^2\frac{\partial^2u}{\partial t^2}+(1-z^2)\frac{\partial^2u}{\partial z^2}+(2t+2)\frac{\partial u}{\partial t}-2z\frac{\partial u}{\partial z}=0. \quad (13)$$

By using the usual separation method $u(t,z)=v(t)w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the equations

$$t^2d^2v/dt^2+(2t+2)dv/dt-sv=0 \quad (14)$$

and

$$(1-z^2)d^2w/dz^2-2zdw/dz+sw=0, \quad (15)$$

where s is a separation parameter. Writing $s=n(n+$

1), the solutions of (14) which are regular near $t=0$ are represented for $n=0,1,2,\cdots$ by the Bessel polynomials

$$v(t)=y_n(t):=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}(t/2)^k,$$

while (15) has for the same values of s the Legendre polynomial solution $w(z)=P_n^{(0,0)}(z):=P_n(z)$, and hence (13) have polynomial solutions $u(t,z)=y_n(t)\cdot P_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result characterizes all entire solutions of (13).

Theorem 3 The partial differential equation (13) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}\frac{c_n}{n!}y_n(t)P_n(z) \tag{16}$$

such that

$$\limsup_{n\to\infty}|c_n|^{1/n}=0. \tag{17}$$

Proof Let $f(t,z)$ be an entire solution of (13). For any $t\in\mathbf{C}$, the entire function $f_t(z)=f(t,z)$ on $z\in\mathbf{C}$ has a Neumann expansion [15-18] $f_t(z)=\sum_{n=0}^{\infty}v_n(t)\cdot P_n(z)$ such that $\limsup_{n\to\infty}|v_n(t)|^{1/n}=0$.

Since $f(t,z)$ is a solution of (13), we find

$$0=t^2\frac{\partial^2 f}{\partial t^2}+(1-z^2)\frac{\partial^2 f}{\partial z^2}+(2t+2)\frac{\partial f}{\partial t}-2z\frac{\partial f}{\partial z}=$$

$$\sum_{n=0}^{\infty}\left\{\left(t^2\frac{\mathrm{d}^2v_n(t)}{\mathrm{d}t^2}+(2t+2)\frac{\mathrm{d}v_n(t)}{\mathrm{d}t}-n(n+1)v_n(t)\right)P_n(z)+v_n(t)\left((1-z^2)\frac{\mathrm{d}^2P_n(z)}{\mathrm{d}z^2}-2z\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}+n(n+1)P_n(z)\right)\right\},$$

which means

$$t^2\frac{\mathrm{d}^2v_n(t)}{\mathrm{d}t^2}+(2t+2)\frac{\mathrm{d}v_n(t)}{\mathrm{d}t}-n(n+1)v_n(t)=0.$$

Therefore, there exists constants c_n, d_n such that $v_n(t)=c_ny_n(t)/n!+d_ne^{2/t}y_n(-t)$, where $e^{2/t}y_n(-t)$ is second independent solution of (14) with $s=n(n+1)$. Note that $v_n(t)$ are entire functions, so that $d_n=0$, and hence

$$f_t(z)=\sum_{n=0}^{\infty}\frac{c_n}{n!}y_n(t)P_n(z).$$

Asymptotic properties of Bessel polynomials were considered already in [21-22]. It was shown there that, for fixed $t\neq 0$ and $n\to\infty$,

$$y_n(t)\sim\frac{(2n)!}{2^nn!}t^ne^{1/t}.$$

If one uses Stirling's formula for the factorials, this is seen to be equivalent to

$$y_n(t)\sim\sqrt{2}(2nt/e)^ne^{1/t}.$$

Moreover, for $n>1$,

$$\left|y_n(t)-\frac{(2n)!}{2^nn!}t^ne^{1/t}\right|\leqslant K_n(t)\left|\frac{(2n)!}{2^nn!}t^ne^{1/t}\right|,$$

where

$$K_n(t)=\frac{1}{4(n-1)}\left|\frac{1}{t^2}e^{1/|t|-1/t}\right|.$$

Therefore

$$\limsup_{n\to\infty}\sqrt[n]{c_ny_n(t)/n!}=2|t|\limsup_{n\to\infty}\sqrt[n]{|c_n|}.$$

Thus (17) follows.

Conversely, it is easy to check that the entire function defined by (16) satisfies (13).

2 Series Expansions Involving Bessel Polynomials

We consider the following partial differential equation

$$t^2\frac{\partial^2 u}{\partial t^2}-z^2\frac{\partial^2 u}{\partial z^2}+(2t+2)\frac{\partial u}{\partial t}-(2z+2)\frac{\partial u}{\partial z}=0. \tag{18}$$

By using the usual separation method $u(t,z)=v(t)\cdot w(z)$, we easily find that (18) have polynomial solutions $u(t,z)=y_n(t)y_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result characterizes all entire solutions of (18).

Theorem 4 The partial differential equation (18) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}\frac{c_n}{(n!)^2}y_n(t)y_n(z) \tag{19}$$

such that

$$\limsup_{n\to\infty}|c_n|^{1/n}=0. \tag{20}$$

Proof Let $f(t,z)$ be an entire solution of (18). For any $t\in\mathbf{C}$, the entire function $f_t(z)=f(t,z)$ on $z\in\mathbf{C}$ has a Neumann expansion [16,23]

$$f_t(z)=\sum_{n=0}^{\infty}\frac{v_n(t)}{n!}y_n(z)$$

such that $\limsup_{n\to\infty}|v_n(t)|^{1/n}=0$.

Since $f(t,z)$ is a solution of (18), according to the method in Section 1, we find that $v_n(t)$ satisfy

$$t^2\frac{d^2v_n(t)}{dt^2}+(2t+2)\frac{dv_n(t)}{dt}-n(n+1)v_n(t)=0.$$

Therefore, there exists constants c_n, d_n such that $v_n(t)=c_n y_n(t)/n! + d_n e^{2/t} y_n(-t)$. Note that $v_n(t)$ are entire functions, so that $d_n=0$, and hence

$$f_t(z)=\sum_{n=0}^{\infty}\frac{c_n}{(n!)^2}y_n(t)y_n(z),$$

so that (20) follows from the arguments in proof of Theorem 3.

Conversely, it is easy to check that the entire function defined by (19) satisfies (18).

3 Series Expansions Involving Chebyshev Polynomials

Chebyshev polynomials $T_n(z)$ of the first kind are defined by

$$T_n(z)=\frac{n!}{(1/2)_n}P_n^{(-1/2,-1/2)}(z),$$

which satisfy the following differential equations

$$(1-z^2)d^2w/dz^2-zdw/dz+n^2w=0.$$

3.1 Products of Chebyshev Polynomials

We consider the following partial differential equation

$$(1-t^2)\frac{\partial^2u}{\partial t^2}-(1-z^2)\frac{\partial^2u}{\partial z^2}-t\frac{\partial u}{\partial t}+z\frac{\partial u}{\partial z}=0. \quad (21)$$

By using the usual separation method $u(t,z)=v(t)\cdot w(z)$, we easily find that (21) have polynomial solutions $u(t,z)=T_n(t)T_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result characterizes all entire solutions of (21).

Theorem 5 The partial differential equation (21) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}c_nT_n(t)T_n(z)$$

such that $\limsup_{n\to\infty}|c_n|^{1/n}=0$.

Proof Similar to the proof of Theorem 2, we can prove this result.

3.2 Products of Chebyshev Polynomials and Trigonometric Functions

We consider the following partial differential equation

$$\partial^2u/\partial t^2-(1-z^2)\partial^2u/\partial z^2+z\partial u/\partial z=0 \quad (22)$$

and characterizes all entire solutions of (22) as follows:

Theorem 6 The partial differential equation (22) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=a_0+b_0t+\sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt))T_n(z). \quad (23)$$

Proof Let $f(t,z)$ be an entire solution of (22). For any $t\in\mathbf{C}$, the entire function $f_t(z)=f(t,z)$ on $z\in\mathbf{C}$ has a Neumann expansion [15-18] $f_t(z)=\sum_{n=0}^{\infty}v_n(t)\cdot T_n(z)$. Since $f(t,z)$ is a solution of (22), we find

$$0=\frac{\partial^2f}{\partial t^2}-(1-z^2)\frac{\partial^2f}{\partial z^2}+z\frac{\partial f}{\partial z}=\sum_{n=0}^{\infty}\left\{\left(\frac{d^2v_n(t)}{dt^2}+n^2v_n(t)\right)T_n(z)-v_n(t)\left((1-z^2)\frac{d^2T_n(z)}{dz^2}-z\frac{dT_n(z)}{dz}+n^2T_n(z)\right)\right\},$$

which means

$$d^2v_n(t)/dt^2+n^2v_n(t)=0.$$

According to basic theory of ordinary differential equations, there exist two constants a_n, b_n such that

$$v_n(t)=\begin{cases}a_0+b_0t, & n=0,\\ a_n\cos(nt)+b_n\sin(nt), & n\geqslant 1.\end{cases}$$

Thus we obtain the expansion (23).

Conversely, it is easy to check that the entire function defined by (23) satisfies (22).

3.3 Products of Chebyshev Polynomials and Bessel Functions

We consider the following partial differential equation

$$t^2\frac{\partial^2u}{\partial t^2}+(1-z^2)\frac{\partial^2u}{\partial z^2}+t\frac{\partial u}{\partial t}-z\frac{\partial u}{\partial z}+t^2u=0. \quad (24)$$

By using the usual separation method $u(t,z)=v(t)\cdot w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the equations

$$t^2d^2v/dt^2+tdv/dt+(t^2-s)v=0 \quad (25)$$

and

$$(1-z^2)d^2w/dz^2-zdw/dz+sw=0, \quad (26)$$

where s is a separation parameter. Writing $s=n^2$, the solutions of (25) which are regular near $t=0$ are represented for $n=0,1,2,\cdots$ by the Bessel functions of the first kind $v(t)=J_n(t)$, while (26) has for the same values of s the Chebyshev polynomial solution

$w(z)=T_n(z)$, and hence (24) have solutions $u(t,z)=J_n(t)T_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result characterizes all entire solutions of (24).

Theorem 7 The partial differential equation (24) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty} n!\,c_n J_n(t)T_n(z) \tag{27}$$

such that

$$\limsup_{n\to\infty}|c_n|^{1/n}=0. \tag{28}$$

Proof Let $f(t,z)$ be an entire solution of (24). For any $t\in\mathbf{C}$, the entire function $f_t(z)=f(t,z)$ on $z\in\mathbf{C}$ has a Neumann expansion[15-18] $f_t(z)=\sum_{n=0}^{\infty} v_n(t)\cdot T_n(z)$ such that

$$\limsup_{n\to\infty}|v_n(t)|^{1/n}=0. \tag{29}$$

Since $f(t,z)$ is a solution of (24), we find

$$0=t^2\frac{\partial^2 f}{\partial t^2}+(1-z^2)\frac{\partial^2 f}{\partial z^2}+t\frac{\partial f}{\partial t}-z\frac{\partial f}{\partial z}+t^2 f=$$

$$\sum_{n=0}^{\infty}\left\{\left(t^2\frac{\mathrm{d}^2 v_n(t)}{\mathrm{d}t^2}+t\frac{\mathrm{d}v_n(t)}{\mathrm{d}t}+(t^2-n^2)v_n(t)\right)T_n(z)+v_n(t)\left((1-z^2)\frac{\mathrm{d}^2T_n(z)}{\mathrm{d}z^2}-z\frac{\mathrm{d}T_n(z)}{\mathrm{d}z}+n^2T_n(z)\right)\right\},$$

which means

$$t^2\mathrm{d}^2v_n(t)/\mathrm{d}t^2+t\mathrm{d}v_n(t)/\mathrm{d}t+(t^2-n^2)v_n(t)=0.$$

According to basic theory of ordinary differential equations, there exist two constants c_n, d_n such that $v_n(t)=n!\ c_nJ_n(t)+d_nN_n(t)$, where $N_n(t)$ is the second kind of Bessel function (Neumann function) of order n. This equation yields easily $d_n=0$ by studying the singularity at $t=0$. Further, according to the arguments in [11], we know that (29) is equivalent to (28). Thus we obtain the expansion (27).

Conversely, it is easy to check that the entire function defined by (27) satisfies (24).

4 Series Expansions Involving Bessel Functions

Carl Neumann introduced a polynomial of degree $n+1$ in $1/t$ as follows

$$O_0(t)=1/t,O_n(t)=\frac{n}{4}\sum_{k=0}^{n/2}\frac{(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n+1-2k},$$

$n\geqslant 1$

called the Neumann′s polynomial of order n. They have the generating function

$$1/(t-z)=O_0(t)J_0(z)+2\sum_{n=0}^{\infty}O_0(t)J_n(z),$$

where J_n are Bessel functions of the first kind. Then the following fact follows easily from the Cauchy formula:

Lemma 1 (Neumann) If $f(z)$ is an analytic function in a closed disc with centre at the coordinate origin, z is an interior point and C denotes the boundary of the disc, then $f(z)=\sum_{n=0}^{\infty} n!\,c_nJ_n(z)$, where

$$c_0=f(0),c_n=\frac{1}{\pi i n!}\int_C O_n(t)f(t)\,\mathrm{d}t.$$

4.1 Products of Bessel Functions

We consider the following partial differential equation

$$t^2\frac{\partial^2 u}{\partial t^2}-z^2\frac{\partial^2 u}{\partial z^2}+t\frac{\partial u}{\partial t}-z\frac{\partial u}{\partial z}+(t^2-z^2)u=0 \tag{30}$$

and characterizes all entire solutions of (30) as follows:

Theorem 8 The partial differential equation (30) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}(n!)^2c_nJ_n(t)J_n(z) \tag{31}$$

such that $\limsup_{n\to\infty}|c_n|^{1/n}=0$.

Proof Let $f(t,z)$ be an entire solution of (30). For any $t\in\mathbf{C}$, Lemma 1 implies that the entire function $f_t(z)=f(t,z)$ on $z\in\mathbf{C}$ has a Neumann expansion $f_t(z)=\sum_{n=0}^{\infty}n!v_n(t)J_n(z)$ such that $\limsup_{n\to\infty}|v_n(t)|^{1/n}=0$.

According to the method in Section 1, we find that $v_n(t)$ satisfy

$$t^2\mathrm{d}^2v_n(t)/\mathrm{d}t^2+t\mathrm{d}v_n(t)/\mathrm{d}t+(t^2-n^2)v_n(t)=0.$$

Therefore, there exists constants c_n, d_n such that $v_n(t)=n!\ c_nJ_n(t)+d_nN_n(t)$. Note that $v_n(t)$ are entire functions, so that $d_n=0$, and hence (31) follows.

Conversely, it is easy to check that the entire function defined by (31) satisfies (30).

4.2 Products of Bessel Functions and Trigonometric Functions

We consider the following partial differential equation

$$\partial^2u/\partial t^2+z^2\partial^2u/\partial z^2+z\partial u/\partial z+z^2u=0 \tag{32}$$

and characterizes all entire solutions of (32) as follows:

Theorem 9 The partial differential equation (32) has an entire solution $f(t,z)$ on $\mathbf{C}^2$ if and only if $f(t,z)$ is an entire function on $\mathbf{C}^2$ expressed by

$$f(t,z) = (a_0 + b_0 t) J_0(z) + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) J_n(z).$$

Proof See Section 3.2.

For example, the equation (32) has an entire solution as follows:

$$\cos(z\cos t) = J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2nt).$$

5 Series Expansions Involving Laguerre (or Hermite) Polynomials

O. Perron [24] has studied in details the asymptotic properties of the confluent hypergeometric function $\Phi(a,c;z)$ as z or one of the parameters a,c tends to infinity. Using his general results and also the relation

$$L_n(\beta,z) = \frac{(\beta+1)_n}{n!}\Phi(-n,\beta+1;z),$$

one can derive asymptotic formulas for Laguerre polynomials $L_n(\beta,z)$ in the region $\mathbf{C}-[0,\infty)$ and on the ray $[0,\infty)$ [4]. These formulas are sufficient to describe the region of convergence of a series of the kind

$$\sum_{n=0}^{\infty} a_n L_n(\beta,z) \tag{33}$$

basing on

$$\lambda_0 = -\limsup_{n\to\infty}\frac{\ln|a_n|}{2\sqrt{n}}. \tag{34}$$

Proposition 1 [17] The quantity λ_0 defined by (34) has the following properties:

(i) if $\lambda_0 \leqslant 0$, the series (33) is divergent at every point of the region $\mathbf{C}-[0,\infty)$;

(ii) if $0<\lambda_0\leqslant\infty$, the series (33) is absolutely uniformly convergent on every compact subset of the region $\Delta(\lambda_0)=\{z\in\mathbf{C}\mid \mathrm{Re}(-z)^{1/2}<\lambda_0\}$ and diverges at every point of the region $\mathbf{C}-\overline{\Delta(\lambda_0)}$.

If we select that branch of $z^{1/2}$ for which $(-z)^{1/2}$ is real and positive when $z<0$, then

$$\mathrm{Re}(-z)^{1/2} = \{(r-x)/2\}^{1/2} = \lambda$$

gives the equation $y^2 = 4\lambda^2(x+\lambda^2)$ of the parabola, where $z = x+\mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$.

Theorem 10 [25-26] For $\beta>-1$, a function $f(z)$ of a complex variable z can be expanded into a general Laguerre series (33) at $\Delta(\lambda_0)$ if and only if $f(z)$ is analytic on $\Delta(\lambda_0)$ and there is a positive number $B(\beta,\lambda)$ associated to every λ with $0\leqslant\lambda<\lambda_0$ such that

$$|f(z)| \leqslant B(\beta,\lambda)\exp\{x/2 - |x|^{1/2}(\lambda^2 - (r-x)/2)^{1/2}\},\ z\in\overline{\Delta(\lambda)}. \tag{35}$$

For series in Hermite polynomials $H_n(z)$, i.e. series of the type

$$\sum_{n=0}^{\infty} a_n H_n(z) \tag{36}$$

the following holds

Proposition 2 [17] The quantity

$$\tau_0 = -\limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{n/2}a_n|}{\sqrt{2n+1}}$$

has the following properties:

(i) if $\tau_0\leqslant 0$, the series (36) is divergent at every point of the region $\mathbf{C}-\mathbf{R}$;

(ii) if $0<\tau_0\leqslant\infty$, the series (36) is absolutely uniformly convergent on every compact subset of the region $S(\tau_0)=\{z\in\mathbf{C}\mid |\mathrm{Im}(z)|<\tau_0\}$ and diverges at every point of the region $\mathbf{C}-\overline{S(\tau_0)}$.

The proof is based on the asymptotic formula for Hermite polynomials given by G. Szegö [4]. The problem of expansion of analytic functions into series of Hermite polynomials found a solution in 1940 by E. Hille [27] as follows:

Theorem 11 A complex function f holomorphic in the region $S(\tau_0)$ $(0<\tau_0\leqslant\infty)$ can be represented in this region by a series of type (36) if and only if for every $0\leqslant\tau<\tau_0$, there exists a constant $B(\tau)\geqslant 0$ such that the holomorphic function f in $S(\tau_0)$ satisfies the inequality

$$|f(z)| \leqslant B(\tau)\exp\{x^2/2 - |x|(\tau^2-y^2)^{1/2}\}$$

for $z = x+\mathrm{i}y\in\overline{S(\tau)}$.

5.1 Products of Laguerre Polynomials

We consider the following partial differential equation

$$t^2\frac{\partial^2 u}{\partial t^2} - z\frac{\partial^2 u}{\partial z^2} + (\alpha+1-t)\frac{\partial u}{\partial t} - (\beta+1-z)\frac{\partial u}{\partial z} = 0 \tag{37}$$

with $\alpha,\beta>-1$. By using the usual separation method $u(t,z)=v(t)w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the equations

$$t\mathrm{d}^2v/\mathrm{d}t^2+(\alpha+1-t)\mathrm{d}v/\mathrm{d}t+sv=0 \tag{38}$$

and

$$z\mathrm{d}^2w/\mathrm{d}z^2+(\beta+1-z)\mathrm{d}w/\mathrm{d}z+sw=0, \tag{39}$$

where s is a separation parameter. Writing $s=n$, the solutions of (38) which are regular near $t=0$ are represented for $n=0,1,2,\cdots$ by the Laguerre polynomials $v(t)=L_n(\alpha,t)$, while (39) has for the same values of s the Laguerre polynomial solution $w(z)=L_n(\beta,z)$, and hence (37) have solutions $u(t,z)=L_n(\alpha,t)\cdot L_n(\beta,z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result follows easily.

Theorem 12 The equation (37) have entire solutions $f(t,z)$ on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}c_nL_n(\alpha,t)L_n(\beta,z) \tag{40}$$

such that

$$-\limsup_{n\to\infty}\frac{\ln|c_n|}{\sqrt{n}}=+\infty. \tag{41}$$

Proof From the formula in [4], we conclude that the following limit relation

$$\lim_{n\to\infty}\frac{\ln|L_n(\beta,z)|}{\sqrt{n}}=2\{(r-x)/2\}^{1/2}=2r^{1/2}\sin(\theta/2) \tag{42}$$

holds uniformly in any finite closed region of $\mathbf{C}$ excluding the non-negative real axis. We also have the estimate [4]

$$|L_n(\beta,x)|\leqslant\frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+1)}\exp(x/2)$$

for $x\geqslant 0$. Note that

$$\Gamma(n+\beta+1)/\Gamma(n+1)\sim n^{\beta}.$$

By using Proposition 1, it is easy to check that the condition (41) implies that $g_z(t)=f(t,z)$ is an entire function of $t\in\mathbf{C}$. Symmetrically, we can prove that $f_t(z)=f(t,z)$ also is an entire function of $z\in\mathbf{C}$, so that (40) is an entire solution of (37).

Now a natural question is that does each entire solution $f(t,z)$ of (37) has the estimate (35) for any t? In other words, we suggest the following question:

Question 1 Does each entire solution $f(t,z)$ of (37) has the expansion (40) satisfying (41)?

5.2 Products of Hermite Polynomials and Laguerre Polynomials

We consider the following partial differential equation

$$\frac{\partial^2u}{\partial t^2}-2z\frac{\partial^2u}{\partial z^2}-2t\frac{\partial u}{\partial t}-2(\alpha+1-z)\frac{\partial u}{\partial z}=0 \tag{43}$$

with $\alpha>-1$. By using the usual separation method $u(t,z)=v(t)w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the equations

$$\mathrm{d}^2v/\mathrm{d}t^2-2t\mathrm{d}v/\mathrm{d}t+sv=0 \tag{44}$$

and

$$z\mathrm{d}^2w/\mathrm{d}z^2+(\alpha+1-z)\mathrm{d}w/\mathrm{d}z+sw/2=0 \tag{45}$$

where s is a separation parameter. Writing $s=2n$, the solutions of (44) which are regular near $t=0$ are represented for $n=0,1,2,\cdots$ by the Hermite polynomials $v(t)=H_n(t)$, while (45) has for the same values of s the Laguerre polynomial solution $w(z)=L_n(\alpha,z)$, and hence (43) have solutions $u(t,z)=H_n(t)L_n(\alpha,z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result follows easily.

Theorem 13 The equation (43) have entire solutions $f(t,z)$ on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}c_nH_n(t)L_n(\alpha,z) \tag{46}$$

such that

$$-\limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{n/2}c_n|}{\sqrt{2n+1}}=+\infty. \tag{47}$$

Proof Fix $t\in\mathbf{C}$. We claim

$$-\lim_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}H_n(t)|}{\sqrt{n}}\geqslant 0. \tag{48}$$

In fact, noting

$$H_{2n}(t)=(-2)^n n!\ L_n(-1/2,t^2),$$

$$H_{2n+1}(t)=(-2)^n n!\ \sqrt{2}tL_n(1/2,t^2),$$

then by using (42), we easily obtain

$$-\lim_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}H_n(t)|}{\sqrt{n}}=+\infty$$

for any $t\in\mathbf{C}-\mathbf{R}$. When $t\in\mathbf{R}$, it is known that there exists a constant A such that [4,27-28]

$$|H_n(t)|\leqslant A(2^n n!)^{1/2}\mathrm{e}^{t^2/2}. \tag{49}$$

Thus by using Stirling formula $n!\ \sim n^n\mathrm{e}^{-n}\sqrt{2\pi n}$. we easily obtain

$$\lim_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}H_n(t)|}{\sqrt{n}}\leqslant 0$$

for $t\in\mathbf{R}$, so that the claim (48) is proved completely. Therefore, the condition (47) and (48) yield

$$-\limsup_{n\to\infty}\frac{\ln|c_nH_n(t)|}{\sqrt{n}}=-\limsup_{n\to\infty}\left\{\frac{\ln|(2n/\mathrm{e})^{n/2}c_n|}{\sqrt{n}}+\right.$$

$$\frac{\ln|(2n/\mathrm{e})^{-n/2}H_n(t)|}{\sqrt{n}}\Big\} \geqslant -\limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{n/2}c_n|}{\sqrt{n}} - \limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}H_n(t)|}{\sqrt{n}} \geqslant$$

$$-\sqrt{2}\limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{n/2}c_n|}{\sqrt{2n+1}} = +\infty.$$

Thus it follows from Proposition 1 that (46) is an entire function of $z\in\mathbf{C}$.

Next fix $z\in\mathbf{C}$. The condition (47) and the formula (42) obviously yields

$$-\limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{n/2}c_nL_n(\alpha,z)|}{\sqrt{2n+1}} = +\infty$$

so that Proposition 2 implies that (46) also is an entire function of $t\in\mathbf{C}$.

5.3 Products of Hermite Polynomials

We consider the following partial differential equation

$$\partial^2u/\partial t^2-\partial^2u/\partial z^2-2t\partial u/\partial t+2z\partial u/\partial z=0. \quad (50)$$

By using the usual separation method $u(t,z)=v(t)\cdot w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the Hermite differential equations, and hence (50) have solutions $u(t,z)=H_n(t)H_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result follows easily.

Theorem 14 The equation (50) have entire solutions $f(t,z)$ on $\mathbf{C}^2$ expressed by

$$f(t,z) = \sum_{n=0}^{\infty}c_nH_n(t)H_n(z)$$

such that $-\limsup_{n\to\infty}\ln|(2n/\mathrm{e})^nc_n|/\sqrt{2n+1}=+\infty$.

Proof It follows easily from the estimate (48) and Proposition 2.

6 Series Expansions Involving Weber Functions

6.1 Products of Weber Functions and Laguerre Polynomials

We consider the following partial differential equation

$$\frac{\partial^2u}{\partial t^2}-z\frac{\partial^2u}{\partial z^2}-(\alpha+1-z)\frac{\partial u}{\partial z}+\frac{(2-t^2)u}{4}=0 \quad (51)$$

with $\alpha>-1$. By using the usual separation method $u(t,z)=v(t)w(z)$, then $v(t)$ and $w(z)$ have to satisfy separately the equations

$$\mathrm{d}^2v/\mathrm{d}t^2+(s+1/2-t^2/4)v=0 \quad (52)$$

and

$$z\mathrm{d}^2w/\mathrm{d}z^2+(\alpha+1-z)\mathrm{d}w/\mathrm{d}z+sw=0, \quad (53)$$

where s is a separation parameter. Writing $s=n$, the solutions of (52) which are regular near $t=0$ are represented for $n=0,1,2,\cdots$ by the Weber (or parabolic cylinder) functions

$$v(t)=D_n(t)=(-1)^n\mathrm{e}^{t^2/4}\mathrm{d}^n\mathrm{e}^{-t^2/2}/\mathrm{d}t^n = \mathrm{e}^{-t^2/4}H_n(t/\sqrt{2}),$$

while (53) has for the same values of s the Laguerre polynomial solution $w(z)=L_n(\alpha,z)$, and hence (51) have solutions $u(t,z)=D_n(t)L_n(\alpha,z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result follows easily.

Theorem 15 The equation (51) have entire solutions $f(t,z)$ on $\mathbf{C}^2$ expressed by

$$f(t,z) = \sum_{n=0}^{\infty}c_nD_n(t)L_n(\alpha,z) \quad (54)$$

such that

$$-\limsup_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{n/2}c_n|}{\sqrt{2n+1}} = +\infty. \quad (55)$$

Proof Fix $t\in\mathbf{C}$. We claim

$$-\lim_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}D_n(t)|}{\sqrt{n}}\geqslant 0. \quad (56)$$

In fact, noting

$$D_{2n}(t)=(-2)^nn!\ \mathrm{e}^{-t^2/4}L_n(-1/2,t^2/2),$$

$$D_{2n+1}(t)=(-2)^nn!\ \sqrt{2}t\mathrm{e}^{-t^2/4}L_n(1/2,t^2/2),$$

then by using (42), we easily obtain

$$-\lim_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}D_n(t)|}{\sqrt{n}} = +\infty$$

for any $t\in\mathbf{C}-\mathbf{R}$. When $t\in\mathbf{R}$, the estimate (49) yields

$$|D_n(t)|\leqslant A(2^nn!)^{1/2}.$$

Thus by using Stirling formula $n!\ \sim n^n\mathrm{e}^{-n}\sqrt{2\pi n}$, we easily obtain

$$\lim_{n\to\infty}\frac{\ln|(2n/\mathrm{e})^{-n/2}D_n(t)|}{\sqrt{n}}\leqslant 0$$

for $t\in\mathbf{R}$, so that the claim (56) is proved completely. Therefore, the condition (55) and (56) yield

$$-\limsup_{n\to\infty}\frac{\ln|c_nD_n(t)|}{\sqrt{n}} =$$

$$-\limsup_{n\to\infty}\Big\{\frac{\ln|(2n/\mathrm{e})^{n/2}c_n|}{\sqrt{n}}+\frac{\ln|(2n/\mathrm{e})^{-n/2}D_n(t)|}{\sqrt{n}}\Big\} \geqslant$$

$$-\limsup_{n\to\infty}\frac{\ln|(2n/e)^{n/2}c_n|}{\sqrt{n}}-$$
$$\limsup_{n\to\infty}\frac{\ln|(2n/e)^{-n/2}D_n(t)|}{\sqrt{n}}\geqslant$$
$$-\sqrt{2}\limsup_{n\to\infty}\frac{\ln|(2n/e)^{n/2}c_n|}{\sqrt{2n+1}}=+\infty.$$

Thus it follows from Proposition 1 that (54) is an entire function of $z\in\mathbf{C}$.

Next fix $z\in\mathbf{C}$. The condition (55) and the formula (42) obviously yields

$$-\limsup_{n\to\infty}\frac{\ln|(2n/e)^{n/2}c_nL_n(\alpha,z)|}{\sqrt{2n+1}}=+\infty$$

so that Proposition 2 implies that (54) also is an entire function of $t\in\mathbf{C}$.

6.2 Products of Hermite Polynomials and Weber Functions

We consider the following partial differential equation

$$\partial^2u/\partial t^2-2\partial^2u/\partial z^2-2t\partial u/\partial t+(z^2-2)u/2=0. \quad (57)$$

By using the usual separation method $u(t,z)=v(t)\cdot w(z)$, we easily find that (57) have entire solutions $u(t,z)=H_n(t)D_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result follows easily from Theorem 14.

Theorem 16 The equation (57) have entire solutions $f(t,z)$ on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}c_nH_n(t)D_n(z)$$

such that

$$-\limsup_{n\to\infty}\frac{\ln|(2n/e)^{n}c_n|}{\sqrt{2n+1}}=+\infty.$$

6.3 Products of Weber Functions

We consider the following partial differential equation

$$\partial^2u/\partial t^2-\partial^2u/\partial z^2+(z^2-t^2)u/4=0. \quad (58)$$

By using the usual separation method $u(t,z)=v(t)\cdot w(z)$, we easily find that (58) have entire solutions $u(t,z)=D_n(t)D_n(z)$ for each $n=0,1,2,\cdots$. Generally speaking, the following result follows easily from Theorem 14.

Theorem 17 The equation (58) have entire solutions $f(t,z)$ on $\mathbf{C}^2$ expressed by

$$f(t,z)=\sum_{n=0}^{\infty}c_nD_n(t)D_n(z)$$

such that

$$-\limsup_{n\to\infty}\frac{\ln|(2n/e)^{n}c_n|}{\sqrt{2n+1}}=+\infty.$$

Similar to Question 1, we may ask that are the conditions in Theorem 13 ~ Theorem 17 necessary?

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2 阶齐次线性偏微分方程与特殊函数乘积关联的整函数解

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摘要: 利用高维值分布理论、特殊函数理论以及经典的特殊常微分方程，研究了几个 2 阶齐次线性偏微分方程，给出了这些偏微分方程与特殊函数乘积密切相关的整函数解的特征，开辟了偏微分方程研究的新途径.

关键词: 2 阶齐次线性偏微分方程; 特殊函数; 整函数解

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