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无界域上具有乘积噪声的强阻尼非自治随机波动方程的吸引子存在性

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摘要: 该文研究无界域上带有强阻尼和乘积噪声的非自治随机波动方程吸引子,利用变换系统的方法对解进行一致估计,并通过解的分解及估计得到所对应系统是拉回渐近紧的,最终可得出原系统存在随机吸引子.

关键词: 波动方程; 随机吸引子; 渐近紧性; 强阻尼; 无界区域
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0 引言

本文研究定义在 $\mathbf{R}^n$ 上带有强阻尼和乘积噪声的非自治随机波动方程的动力学行为:

$$u_{tt}-\Delta u-\mu\Delta u_t+h(u)u_t+\lambda u+f(x,u)=g(x,t)+\sum_{j=1}^{k}b_ju\circ \mathrm{d}\omega_j/\mathrm{d}t,t>\tau,\quad(1)$$

初值为

$$u(\tau,x)=u_0(x),u_t(\tau,x)=u_1(x),\tau\in\mathbf{R},\quad(2)$$

其中 $x\in\mathbf{R}^n(1\leqslant n\leqslant 3)$,$\mu$ 和 λ 是正常数,$g\in L^2_{loc}(\mathbf{R},L^2(\mathbf{R}^n))$,$f(x,u)$ 是满足一定增长条件和耗散性的光滑非线性项. $\{\omega_j\}_{j=1}^k$ 是定义在概率空间 $(\Omega,\mathscr{F},\mathscr{P})$ 上双边实值 Wiener 过程,$\mathscr{F}$是 Borel σ-代数,$\mathscr{P}$是相应的 Wiener 测度,$b_j\in\mathbf{R}(j=1,2,\cdots,k)$,且满足 $\sup\limits_{1\leqslant j\leqslant k}b_j=\hat{b}\leqslant 1$,"$\circ$"表示随机项是在 Stratonovich 意义下的.

假设非线性函数 h 和 f 满足下列条件: $\forall x\in\mathbf{R}^n,u\in\mathbf{R}$,有

$$|f(x,u)|\leqslant c_1|u|^{\gamma}+\varphi_1(x),\varphi_1(x)\in L^2(\mathbf{R}^n),\quad(3)$$

$$f(x,u)u-c_2F(x,u)\geqslant\varphi_2(x),\varphi_2(x)\in L^1(\mathbf{R}^n),\quad(4)$$

$$F(x,u)\geqslant c_3|u|^{\gamma+1}-\varphi_3(x),\varphi_3(x)\in L^1(\mathbf{R}^n),\quad(5)$$

$$|f_u(x,u)|\leqslant c_4|u|^{\gamma-1}+\varphi_4(x),\varphi_4(x)\in H^1(\mathbf{R}^n),\quad(6)$$

其中常数 $c_i>0(i=1,2,3,4)$,$F(x,u)=\int_0^u f(x,s)\mathrm{d}s$,当 $n=1,2$ 时,$\gamma\geqslant 1$,当 $n=3$ 时,$\gamma\in[1,3)$.

式(2)存在 2 个常数 β_1、β_2,使得

$$0<\beta_1\leqslant h(s)\leqslant\beta_2<\infty,\forall s\in\mathbf{R}.\quad(7)$$

关于非自治随机偏微分方程的动力学行为,文献[1-3]已有很多说明. 关于随机波动方程的吸引子也有很多文献涉及,如文献[4]研究了带有乘积噪声项的波动方程吸引子的存在性; 文献[5]阐述了随机波动方程吸引子的上半连续性问题; 文献[6]研究发现当噪声的强度任意大时,即使非线性项有界,也不能保证吸引子存在. 有关对随机偏微分方程的吸引子研究的其他结果可参见文献[7-13]. 本文的方程(1)是用来描述非线性波的数学模型,其中非线性项和乘积噪声的收敛性问题是在处理过程中的难点和重点.

从以往文献的研究可知: 可加噪声描述的是系统的外界随机干扰对系统的影响,乘积噪声则描述了系统的内部随机干扰对系统产生的影响. 由于可加噪声和乘积噪声对系统的影响存在本质的不同,所以根据噪声项的不同,对随机波动方程的吸引子研究可以分成 2 类. 一类是噪声为可加噪声. 文献[14]证明了当噪声为可加噪声时随机波动方程吸引子的存在性,并且对吸引子的分形维数做出了估计; 文献[15]对一类带有弱阻尼和可加噪声的随机

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波动方程进行了研究,证明了该方程在 $H_0^1(D) \times L^2(D)$ 中存在随机吸引子. 另一类是噪声为乘积噪声. 文献[16]就带有乘积噪声的随机波动方程的吸引子存在性问题进行了研究; 文献[6]研究了带阻尼项和乘积噪声的随机波动方程的随机吸引子,并得出了随机吸引子的存在性条件.

对于一个 Cauchy 问题来说,在无界域上,要对随机波动方程所对应的无穷维动力系统进行研究,最大的难点在于解决 Sobolev 嵌入的紧性问题. 而本文借助在文献[4,16]中的处理方法来克服这个困难,先得出 $\mathbf{R}^n$ 上有界区域外解的一致估计,再对有界区域内的解进行分解,以此得到解的渐近紧性,最后得出该系统存在随机吸引子.

在本文中,$\|\cdot\|$ 表示 $L^2(\mathbf{R}^n)$ 上的范数,$(\cdot,\cdot)$ 表示 $L^2(\mathbf{R}^n)$ 上的内积,$\|\cdot\|_p$ 表示 $L^p(\mathbf{R}^n)$ 空间上的范数,$\|\cdot\|_X$ 表示 Banach 空间上的范数. 字母 c 和 $c_i(i=1,2,\cdots)$ 表示正常数,其值允许有所变化.

1　基本概念和引理

令 X 是可分的 Hilbert 空间,$(\Omega,\mathscr{F},\mathscr{P})$ 是概率空间,其中 $\Omega=\{(\omega_1,\omega_2,\cdots,\omega_k)\in C(\mathbf{R},\mathbf{R}^k):\omega(0)=0\}$,$\mathscr{F}$ 是 Ω 上紧开拓扑诱导的 Borel σ-代数,$\mathscr{P}$ 是 $(\Omega,\mathscr{F})$ 上相应的 Wiener 测度. 定义变换

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\omega\in\Omega,t\in\mathbf{R}, \quad (8)$$

称 $(\Omega,\mathscr{F},\mathscr{P},(\theta_t)_{t\in\mathbf{R}})$ 为度量动力系统.

定义1　若 $\forall\tau\in\mathbf{R},\omega\in\Omega,t,s\in\mathbf{R}^+$,有

(i) $\Phi(\cdot,\tau,\cdot,\cdot):\mathbf{R}^+\times\Omega\times X\to X$ 为 $(\mathscr{B}(\mathbf{R}^+)\times\mathscr{F}\times\mathscr{B}(X),\mathscr{B}(X))$-可测;

(ii) $\Phi(0,\tau,\omega,\cdot)$ 在 X 上为恒同映射;

(iii) $\Phi(t+s,\tau,\omega,\cdot)=\Phi(t,\tau+s,\theta_s\omega,\cdot)\circ\Phi(s,\tau,\omega,\cdot)$;

(iv) $\Phi(t,\tau,\omega,\cdot):X\to X$ 连续,

则称定义在 X 和 $\mathbf{R}$ 以及 $(\Omega,\mathscr{F},\mathscr{P},(\theta_t)_{t\in\mathbf{R}})$ 上的映射 $\Phi:\mathbf{R}^+\times\mathbf{R}\times\Omega\times X\to X$ 为连续随机动力系统.

令 $\mathscr{D}$ 表示 X 上的非空子集族组成的集合.

定义2　若 $\exists x_0\in X$,对所有的 $c>0,\tau\in\mathbf{R},\omega\in\Omega$,满足

$$\lim_{t\to+\infty}\mathrm{e}^{-ct}d(D(\tau+t,\theta_t\omega),x_0)=0.$$

给定 $D\in\mathscr{D}$,称 $\Omega(D)=(\Omega(D,\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega)$ 为 D 的 Ω-极限集,其中

$$\Omega(D,\tau,\omega)=\bigcap_{r\geqslant0}\overline{\bigcup_{t\geqslant r}\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))},$$

则称 X 上的非空子集 $D=(D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega)$ 是缓增的.

定义3　若 $\forall\tau\in\mathbf{R},\omega\in\Omega$,当 $t_n\to+\infty$,$x_n\in D(\tau-t_n,\theta_{-t_n}\omega)$,$\{D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$ 时,序列 $\{\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,x_n\}_{n=1}^{\infty}$ 在 X 中有收敛子列,则称随机动力系统 Φ 在 X 上是 $\mathscr{D}$-拉回渐近紧的.

定义4　若 $\forall\tau\in\mathbf{R},\omega\in\Omega,D\in\mathscr{D}$,$\exists T=T(D,\tau,\omega)>0$,当 $t\geqslant T$ 时,有

$$\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))\subseteq K(\tau,\omega),$$

则称 $K=\{K(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$ 是 Φ 的 $\mathscr{D}$-拉回吸收集.

定义5　若 $\forall t\in\mathbf{R}^+,\tau\in\mathbf{R},\omega\in\Omega$,满足

(i) $\mathscr{A}(\tau,\omega)$ 在 X 上是紧的且 $\mathscr{A}$ 关于 $\mathscr{F}$ 在 Ω 中是可测的;

(ii) $\Phi(t,\tau,\omega,\mathscr{A}(\tau,\omega))=\mathscr{A}(t+\tau,\theta_t\omega)$,即 $\mathscr{A}$ 是不变的;

(iii) $\mathscr{A}$ 吸引 $\mathscr{D}$ 中任意集合,即 $\forall D=\{D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$,有

$$\lim_{t\to\infty}d_H(\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega)),\mathscr{A}(\tau,\omega))=0,$$

则称 $\mathscr{A}=\{\mathscr{A}(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$ 是 Φ 的 $\mathscr{D}$-拉回吸引子,其中 d_H 是 X 上的 Hausdorff 半距离.

引理1[1]　令 $\mathscr{D}$ 是 X 上包含闭的非空子集族. 若 Φ 在 X 上是 $\mathscr{D}$-拉回渐近紧的,同时在 $\mathscr{D}$ 上存在一个闭可测的 $\mathscr{D}$-拉回吸收集 K,则 Φ 在 $\mathscr{D}$ 上存在唯一的 $\mathscr{D}$-拉回吸引子 $\mathscr{A}$.

2　随机波方程对应的随机动力系统

本部分将阐述式(1)~(2)所表示的连续随机动力系统.

首先,令 $\zeta=u_t+\delta u$,其中 δ 是一个充分小的正常数,则将式(1)~(2)转化为

$$\mathrm{d}u/\mathrm{d}t=\zeta-\delta u, \quad (8)$$

$$\mathrm{d}\zeta/\mathrm{d}t-\mu\Delta\zeta+(\mu\delta-1)\Delta u+(h(u)-\delta)\zeta+(\delta^2+\lambda-\delta h(u))u+f(x,u)=g(x,t)+\sum_{j=1}^{k}b_ju\circ\mathrm{d}\omega_j/\mathrm{d}t, \quad (9)$$

初值为

$$u(\tau,x)=u_0(x),\zeta(\tau,x)=\zeta_0(x), \quad (10)$$

其中 $\zeta(x)=u_1(x)+\delta u_0(x)$,$x\in\mathbf{R}^n$.

给定 $\omega\in\Omega$,记

$$z_j(\theta_t\omega_j)=-\xi\int_{-\infty}^{0}\mathrm{e}^{\xi s}(\theta_t\omega_j)(s)\mathrm{d}s,t\in\mathbf{R},\sigma>0,$$

则 $z_j(\theta_t\omega_j)(j=1,2,\cdots,k)$ 是1维 Ornstein-Uhlenbeck 过程. 另外,随机变量 $|z_j(\theta_t\omega_j)|$ 是缓增的且 $z_j(\theta_t\omega_j)$ 是连续的,$z_j(\theta_t\omega_j)$ 满足方程

$$\mathrm{d}z_j(\theta_t\omega_j)+\delta z_j(\theta_t\omega_j)\mathrm{d}t=\mathrm{d}\omega_j(t).$$

接着定义式(8)~(10)所对应的随机动力系统. 令 $z(\theta_t\omega)=\sum_{j=1}^{k}b_jz_j(\omega)$, $v(t)=\zeta(t)-z(\theta_t\omega)u(t)$,则 v 满足

$$\mathrm{d}u/\mathrm{d}t=v-\delta u+z(\theta_t\omega)u(t),\tag{11}$$

$$\mathrm{d}v/\mathrm{d}t-\mu\Delta v+(\mu\delta-\mu z-1)\Delta u+(h(u)-\delta)v+(\delta^2+\lambda-\delta h(u))u+f(x,u)=g(x,t)-zv-(z+h(u)-3\delta)zu,\tag{12}$$

初值为

$$u(\tau,x)=u_0(x),v(\tau,x)=v_0(x),\tag{13}$$

其中 $v_0(x)=\zeta_0(x)-z(\theta_\tau\omega)u_0,x\in\mathbf{R}^n$.

取充分小的 $\delta>0$,使得 $\beta_1-\delta>0,\lambda+\delta^2-\beta_2\delta>0,1-\mu\delta>0$. 令

$$\sigma=\min\{\beta_1-\delta,\lambda+\delta^2-\beta_2\delta,1-\mu\delta\}/2,$$

假设 $g(x,t)$ 满足: $\exists\sigma>0$,使得

$$\int_{-\infty}^{0}\mathrm{e}^{\sigma s}\|g(\cdot,s+\tau)\|^2\mathrm{d}s<\infty,\forall\tau\in\mathbf{R}.\tag{14}$$

由式(14)可得

$$\lim_{k\to\infty}\int_{-\infty}^{0}\int_{|x|\geqslant k}\mathrm{e}^{\sigma s}|g(x,s+\tau)|^2\mathrm{d}x\mathrm{d}s,\forall\tau\in\mathbf{R}.\tag{15}$$

令 $E(\mathbf{R}^n)=H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$,其标准范数为

$$\|Y\|_{H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)}=(\|\nabla u\|^2+\|u\|^2+\|v\|^2)^{1/2},Y=(u,v)\in E(\mathbf{R}^n),$$

可以定义 $E(\mathbf{R}^n)$ 上的等价范数:

$$\|Y\|_{E(\mathbf{R}^n)}=(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+(1-\mu\delta)\|\nabla u\|^2)^{1/2},Y=(u,v)\in E(\mathbf{R}^n).$$

引理2[17] 令 $\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0)=(u(t+\tau,\tau,\theta_{-\tau}\omega,u_0),v(t+\tau,\tau,\theta_{-\tau}\omega,v_0))$,其中 $\varphi_0=(u_0,v_0)$,且满足条件(3)~(7),则在 $E(\mathbf{R}^n)$ 上, $\forall\omega\in\Omega,\tau\in\mathbf{R},\varphi\in E(\mathbf{R}^n)$,式(11)~(13)存在唯一解 $\varphi(\cdot,\tau,\omega,\varphi_0)\in C([\tau,+\infty),E(\mathbf{R}^n))$,其中 $\varphi(\tau,\tau,\omega,\varphi_0)=\varphi_0$,且该解关于初值 φ_0 连续依赖,此时方程(1)~(2)生成连续随机动力系统 $\Phi:\mathbf{R}^+\times\mathbf{R}\times\Omega\times E(\mathbf{R}^n)\to E(\mathbf{R}^n)$,

$$\Phi(t,\tau,\omega,\varphi_0)=\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0),\forall(t,\tau,\omega,\varphi_0)\in\mathbf{R}^+\times\mathbf{R}\times\Omega\times E(\mathbf{R}^n).$$

3 解的一致估计

本部分对式(8)~(10)的解进行一致估计.

引理3 假设条件(3)~(7)和式(14)成立,对所有的 $\tau\in\mathbf{R},\omega\in\Omega$ 和 $D=\{D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$,则 $\exists T=T(\tau,\omega,D)>0$,使得当 $t\geqslant T$ 时,式(11)~(13)的解 (u,v) 满足

$$\|Y(\tau,\tau-t,\theta_{-t}\omega,\varphi_0)\|^2_{E(\mathbf{R}^n)}\leqslant M(1+R(\tau,\omega)),$$

其中 $\varphi_0=(u_0,v_0)\in D(\tau-t,\theta_{-t}\omega)$, $R(\tau,\omega)=\int_{-\infty}^{0}\mathrm{e}^{2\int_0^s N(r,\omega)\mathrm{d}r}(\|g(x,s+\tau)\|^2/(2\delta)+|z(\theta_t\omega)|)\mathrm{d}s$, M 是不依赖于 τ、ω、D 和 ε 的正常数.

证 将式(12)与 v 在 $L^2(\mathbf{R}^n)$ 中作内积,可得

$$\mathrm{d}\|v\|^2/(2\mathrm{d}t)+\mu\|\nabla v\|^2+(h(u)-\delta)\|v\|^2+(\lambda+\delta^2-\delta h(u))(u,v)+(f(x,u),v)-(1-\mu\delta)\cdot(\Delta u,v)=(g(x,t),v)-z\|v\|^2-(z+h-3\delta)z(u,v)+\mu z(\Delta u,v).\tag{16}$$

由式(11)可得

$$(u,v)=\mathrm{d}\|u\|^2/(2\mathrm{d}t)+\delta\|u\|^2-z(\theta_t\omega)\|u\|^2,\tag{17}$$

$$-(\Delta u,v)=\mathrm{d}\|\nabla u\|^2/(2\mathrm{d}t)+\delta\|\nabla u\|^2-z(\theta_t\omega)\|\nabla u\|^2,\tag{18}$$

$$(f(x,u),v)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x+\delta(f(x,u),u)-z(\theta_t\omega)(f(x,u),u).\tag{19}$$

将式(17)~(19)代入式(16),可得

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2+(\lambda+\delta^2-h(u)\delta)\|u\|^2+(1-\mu\delta)\|\nabla u\|^2+2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x)+(h(u)-\delta)\|v\|^2+\delta(\lambda+\delta^2-\delta h(u))\|u\|^2+\delta(1-\mu\delta)\|\nabla u\|^2+\delta(f(x,u),u)+\mu\|\nabla v\|^2\leqslant(\lambda+\delta^2-\delta\beta_1)z(\theta_t\omega)\cdot\|u\|^2+(1-\mu\delta)z(\theta_t\omega)\|\nabla u\|^2+z(\theta_t\omega)(f(x,u),u)-z(\theta_t\omega)\|v\|^2+(3\delta-\beta_1-z)z\|u\|\|v\|+\mu z(\Delta u,v)+(g(x,t),v).\tag{20}$$

由式(3)~(6)可得

$$\delta(f(x,u),u)\geqslant\delta c_2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x+\delta\int_{\mathbf{R}^n}\varphi_2(x)\mathrm{d}x,\tag{21}$$

$$z(\theta_t\omega)(f(x,u),u)\leqslant c_1z(\theta_t\omega)\int_{\mathbf{R}^n}|u|^{\gamma+1}\mathrm{d}x+|z\theta_t\omega|\|\varphi_1(x)\|^2/2+|z(\theta_t\omega)|\|u\|^2/2\leqslant 2c_1\int_{\mathbf{R}^n}(F(x,u)+\varphi_3(x))\mathrm{d}x/c_3+|z(\theta_t\omega)|\|\varphi_1(x)\|^2/2+|z(\theta_t\omega)|\cdot\|u\|^2/2.\tag{22}$$

由 Cauchy-Schwarz 不等式和 Young 不等式可知

$$z(\theta_t\omega)(3\delta-\beta_1-z(\theta_t\omega))\|u\|\|v\|+\mu z(\theta_t\omega)(\Delta u,v)\leqslant|z(\theta_t\omega)|^2(3\delta-\beta_1)(\|u\|^2+\|v\|^2)/2+\mu\|\nabla v\|^2/2+|z(\theta_t\omega)|^2(\|u\|^2+\|v\|^2+\mu\|\nabla u\|^2)/2,\tag{23}$$

$$(g(x,t),v)\leqslant\|g(x,t)\|^2/(4\delta)+\delta\|v\|^2.\tag{24}$$

将式(21)~(24)代入式(20),得

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2+(\lambda+\delta^2-\delta\beta_2)\|u\|^2+(1-$$

$\mu\delta)\|\nabla u\|^2+2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x)+\mu\|\nabla v\|^2/2+(\beta_1-2\delta)\|v\|^2+\delta(\lambda+\delta^2-\delta\beta_2)\|u\|^2+\delta(1-\mu\delta)\|\nabla u\|^2+\delta c_2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x\leqslant|z(\theta_t\omega)|(\|v\|^2+(\lambda+\delta^2-\delta\beta_1+1/2)\|u\|^2+(1-\mu\delta)\|\nabla u\|^2+c_1\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x/c_3)+|z(\theta_t\omega)|(3\delta-\beta_1)(\|u\|^2+\|v\|^2)/2+|z(\theta_t\omega)|^2(\|u\|^2+\|v\|^2+\mu\|\nabla u\|^2)/2+|z(\theta_t\omega)|\|\varphi_1(x)\|^2/2+c_1|z(\theta_t\omega)|\|\varphi_3(x)\|_1/c_3+\delta\|\varphi_2(x)\|_1+\|g(x,t)\|^2/(4\delta).\qquad(25)$

令

$$\begin{cases}\eta_1=\min\{\delta,\delta c_2/2,\beta_1-2\delta\},\\ \eta_2=\max\{3\delta/2+1,(\lambda+\delta^2-\delta\beta_1+1/2+3\delta/2)/(\lambda+\delta^2-\delta\beta_2),c_1/(2c_3)\},\\ \gamma_1=\max\{1,1/(\lambda+\delta^2-\delta\beta_2),\mu/(1-\mu\delta)\},\\ \gamma_2=\beta_1-3\delta.\end{cases}\qquad(26)$$

则式(25)可化为

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2+(\lambda+\delta^2-\delta\beta_2)\|u\|^2+(1-\mu\delta)\|\nabla u\|^2+2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x)+\mu\|\nabla v\|^2/2\leqslant-(\eta_1-\eta_2|z(\theta_t\omega)|-\gamma_1(|z(\theta_t\omega)|^2/2+\gamma_2\cdot|z(\theta_t\omega)|/2))(\|v\|^2+(\lambda+\delta^2-\delta\beta_2)\|u\|^2+(1-\mu\delta)\|\nabla u\|^2+2\int_{\mathbf{R}^n}F(x,u)\mathrm{d}x)+|z(\theta_t\omega)|\cdot\|\varphi_1(x)\|^2/2+c_1|z(\theta_t\omega)|\|\varphi_3(x)\|_1/c_3+\delta\|\varphi_2(x)\|_1+\|g(x,t)\|^2/(4\delta)+\gamma_1|z(\theta_t\omega)|^2\|\varphi_3(x)\|_1+\gamma_1\gamma_2|z(\theta_t\omega)|\|\varphi_3(x)\|_1.\qquad(27)$$

记

$$N(\tau,\omega)=\eta_1-\eta_2|z(\theta_t\omega)|-\gamma_1(|z(\theta_2\omega)|^2/2+\gamma_2|z(\theta_t\omega)|/2).\qquad(28)$$

在 $[\tau-t\tau,\tau]$ 上对式(27)运用 Gronwall 不等式得

$$\|v(\tau,\tau-t,\omega,v_0)\|^2+(\lambda+\delta^2-\delta\beta_2)\|u(\tau,\tau-t,\omega,u_0)\|^2+(1-\mu\delta)\|\nabla u(\tau,\tau-t,\omega,u_0)\|^2+2\int_{\mathbf{R}^n}F(x,u(\tau,\tau-t,\omega,u_0))\mathrm{d}x+\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_\tau^s N(s,\omega)\mathrm{d}s}\cdot\|\nabla v\|^2\mathrm{d}s\leqslant\mathrm{e}^{2\int_\tau^{\tau-t}N(s,\omega)\mathrm{d}s}(\|v_0\|^2+(\lambda+\delta^2-\delta\beta_2)\cdot\|u_0\|^2+(1-\mu\delta)\|\nabla u_0\|^2+2\int_{\mathbf{R}^n}F(x,u_0)\mathrm{d}x)+\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_\tau^s N(r,\omega)\mathrm{d}r}(\|g(x,s)\|^2/(2\delta)+2c_1|z(\theta_s\omega)|\cdot\|\varphi_3(x)\|_1/c_3+2\delta\|\varphi_2(x)\|_1+|z(\theta_s\omega)|\|\varphi_1(x)\|^2+2\gamma_1|z(\theta_s\omega)|^2\|\varphi_3(x)\|_1+2\gamma_1\gamma_2|z(\theta_s\omega)|\cdot\|\varphi_3(x)\|_1)\mathrm{d}s.\qquad(29)$$

用 $\theta_{-\tau}\omega$ 代替 ω,则由式(29)可得

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\delta\beta_2)\cdot\|u(\tau,\tau-1,\theta_{-\tau}\omega,u_0)\|^2+(1-\mu\delta)\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+2\int_{\mathbf{R}^n}F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0))\mathrm{d}x+\int_{-t}^{0}\mathrm{e}^{2\int_0^s N(s,\omega)\mathrm{d}s}\|\nabla v\|^2\mathrm{d}s\leqslant\mathrm{e}^{2\int_0^{-t}N(s,\omega)\mathrm{d}s}(\|v_0\|^2+(\lambda+\delta^2-\delta\beta_2)\|u_0\|^2+(1-\mu\delta)\|\nabla u_0\|^2+2\int_{\mathbf{R}^n}F(x,u_0)\mathrm{d}x)+\int_{-t}^{0}\mathrm{e}^{2\int_0^s N(r,\omega)\mathrm{d}r}(2c_1|z(\theta_s\omega)|\|\varphi_3(x)\|_1/c_3+|z(\theta_s\omega)|\|\varphi_1(x)\|^2+2\gamma_1|z(\theta_s\omega)|^2\|\varphi_3(x)\|_1+2\delta\|\varphi_2(x)\|_1+2\gamma_1\gamma_2|z(\theta_s\omega)|\|\varphi_3(x)\|_1)\mathrm{d}s+\int_{-t}^{0}\mathrm{e}^{2\int_0^s N(r,\omega)\mathrm{d}r}\|g(x,s+\tau)\|^2/(2\delta)\mathrm{d}s.$$

注意到 $z_j(\theta_t\omega)(j=1,2,\cdots,k)$ 具有稳定性和遍历性,则由 Birkhoff 遍历定理可得

$$\lim_{t\to\infty}\int_{-t}^{0}|z_j(\theta_t\omega)|\mathrm{d}r/t=E(|z_j(\theta_t\omega)|)=1/\sqrt{\pi\delta},$$

$$\lim_{t\to\infty}\int_{-t}^{0}|z_j(\theta_t\omega)|^2\mathrm{d}r/t=E(|z_j(\theta_t\omega)|^2)=1/(2\delta).$$

所以,$\exists T_1(\tau,\omega,D)>0$,当 $t\geqslant T_1(\tau,\omega,D)$ 时,使得 $\int_{-t}^{0}|z(\theta_r\omega)|\mathrm{d}r<2t/\sqrt{\pi\delta}$,$\int_{-t}^{0}|z(\theta_r\omega)|^2\mathrm{d}r<t/\delta$.

令 b_0 为常数,满足

$$b_0=\min\{1,\eta_1\delta\sqrt{\pi\delta}/(4\eta_2\sqrt{k}\delta+\gamma_1 k\sqrt{\pi\delta}+2\gamma_1\gamma_2\sqrt{k}\delta)\},$$

则 $\forall\hat{b}\leqslant b_0$,当 $s<T_1(\tau,\omega,D)$ 时,有 $\mathrm{e}^{2\int_0^t N(\tau,\omega)\mathrm{d}\tau}\leqslant\mathrm{e}^{\sigma s}$,从而得到

$$R(\tau,\omega)=\int_{-t}^{0}\mathrm{e}^{2\int_0^s N(r,\omega)\mathrm{d}r}(2c_1|z(\theta_s\omega)|\|\varphi_3(x)\|_1/c_3+|z(\theta_s\omega)|\|\varphi_1(x)\|^2+2\gamma_1|z(\theta_s\omega)|^2\|\varphi_3(x)\|_1+2\delta\|\varphi_2(x)\|+2\gamma_1\gamma_2|z(\theta_s\omega)|\|\varphi_3(x)\|_1)\mathrm{d}s+\int_{-\infty}^{0}\mathrm{e}^{2\int_0^s N(r,\omega)\mathrm{d}r}\|g(x,s+\tau)\|^2/(2\delta)\mathrm{d}s.$$

由 $|z(\theta_t\omega)|$ 稳定性易得

$$\int_{-t}^{0}\mathrm{e}^{2\int_0^s N(r,\omega)\mathrm{d}s}\|\nabla v\|^2\mathrm{d}s\leqslant c(1+R(\tau,\omega)).$$

由 $\varphi_1(x)\in L^2(\mathbf{R}^n)$,$\varphi_2(x)\in L^1(\mathbf{R}^n)$,$\varphi_3(x)\in L^1(\mathbf{R}^n)$ 和 $|z(\theta_t\omega)|$ 的稳定性可得积分 $R(\tau,\omega)$ 收敛.

当 $(u_0,v_0)\in D(\tau-t,\theta_{-t}\omega)$ 时,可得

$\lim_{t\to\infty}(\|v_0\|^2+(\lambda+\delta^2-\delta\beta_2)\|u_0\|^2+(1-$

$\mu\delta)\|\nabla u_0\|^2+2\int_{\mathbf{R}^n}F(x,u_0)\,dx)=0.$

并且,由式(3)和式(4)可得

$$\int_{\mathbf{R}^n}F(x,u_0)\,dx\leqslant c(1+\|u_0\|^2+\|u_0\|_{H^1}^{\gamma+1}).$$

$\exists T=T(\tau,\omega,D)>T_1(\tau,\omega,D)$,当 $t>T$ 时,满足

$$\|v(\tau,\tau-1,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\delta\beta_2)\cdot\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0\|^2+(1-\mu\delta)\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\leqslant M(1+R(\tau,\omega)).$$

因此,引理3得证.

由引理3可得

引理4 假设条件(3)~(7)和式(14)成立,则定义在 $E(\mathbf{R}^n)$ 上的连续随机动力系统 Φ 有吸收集 $D=\{D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$,其中 $D(\tau,\omega)$ 定义为

$$D(\tau,\omega)=\{(u,v)\in E(\mathbf{R}^n):\|(u,v)\|_{E(\mathbf{R}^n)}^2\leqslant M(1+R(\tau,\omega))\}.$$

若令 ρ 是光滑函数,并假设对所有的 $s\in\mathbf{R}$,有 $0\leqslant\rho(s)\leqslant1$,同时令

$$\rho(s)=\begin{cases}0,0\leqslant s\leqslant1,\\1,|s|\geqslant2,\end{cases}$$

则存在正常数 κ_1,κ_2,满足对所有的 $s\in\mathbf{R}$,有 $|\rho'(s)|\leqslant\kappa_1$ 以及 $|\rho''(s)|\leqslant\kappa_2$. 给定 $r\geqslant1$,记 $B_r=\{x\in\mathbf{R}^n:|x|\leqslant r\}$ 以及 B_r 的补集 $\mathbf{R}^n\backslash B_r$.

为了得到渐近紧性,需要如下结论.

引理5 假设条件(3)~(7)和式(14)成立,记 $\{B(\omega)\}\in\mathscr{D}$,$\varphi_0(\omega)\in B(\omega)$. 令 $\varepsilon>0$,$\tau\in\mathbf{R}$,$\omega\in\Omega$,$D=\{D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$,则 $\exists\hat{T}=\hat{T}(\tau,\omega,D,\varepsilon)>0$,$\hat{R}=\hat{R}(\tau,\omega,\varepsilon)\geqslant1$,使得当 $t\geqslant\hat{T}$,$r\geqslant\hat{R}$ 时,

$$\int_{\mathbf{R}^n\backslash B_r}(|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\nabla u|^2+|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2\,dx\leqslant\varepsilon.$$

证 将式(12)与 $\rho(|x|^2/r^2)v$ 在 $L^2(\mathbf{R}^n)$ 中作内积,可得

$$\frac{1}{2}\cdot\frac{d}{dt}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|v|^2dx-\mu\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot v\Delta v dx-(1-\mu\delta)\int_{\mathbf{R}^n}(|x|^2/r^2)v\Delta u dx+(h(u)-\delta)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|v|^2dx+(\lambda+\delta^2-\delta h(u))\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)uv dx+\int_{\mathbf{R}^n}\rho(|x|^2/r^2)f(x,u)v dx=\int_{\mathbf{R}^n}\rho(|x|^2/r^2)g(x,t)v dx-z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot|v|^2dx-(z(\theta_t\omega)+h(u)-3\delta)z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot uv dx+\mu z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)v\Delta u dx. \tag{30}$$

由式(11)得到

$$-\int_{\mathbf{R}^n}\rho(|x|^2/r^2)v\Delta u dx=-\frac{2x}{r^2}\int_{\mathbf{R}^n}\nabla u\rho'(|x|^2/r^2)v dx+\frac{1}{2}\cdot\frac{d}{dt}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\nabla u|^2dx+\delta\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\nabla u|^2dx-z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\nabla u|^2dx, \tag{31}$$

$$\int_{\mathbf{R}^n}\rho(|x|^2/r^2)f(x,u)v dx=\frac{d}{dt}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot F(x,u)dx+\delta\int_{\mathbf{R}^n}\rho(|x|^2/r^2)f(x,u)u dx-z(\theta_t\omega)\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)f(x,u)u dx, \tag{32}$$

$$\int_{\mathbf{R}^n}u\rho(|x|^2/r^2)v dx=\frac{1}{2}\cdot\frac{d}{dt}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|u|^2dx+\delta\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|u|^2dx-z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot|u|^2dx. \tag{33}$$

根据假设条件(3)~(6)和 Young 不等式,有

$$\int_{\mathbf{R}^n}g(x,t)\rho(|x|^2/r^2)v dx\leqslant\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot(|g(x,t)|^2/(4\delta)+\delta\|v\|^2)dx, \tag{34}$$

$$\int_{\mathbf{R}^n}\rho(|x|^2/r^2)f(x,u)u dx\geqslant c_2\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot F(x,u)dx+\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\varphi_2(x)|dx, \tag{35}$$

$$z(\theta_t\omega)\int_{\mathbf{R}^n}f(x,u)\rho(|x|^2/r^2)u dx\leqslant|z(\theta_t\omega)|\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|\varphi_1(x)|^2+|u|^2)dx/2+c_1\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|u|^{\gamma+1}dx\leqslant c_1|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(F(x,u)+\varphi_3(x))dx/c_3+|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|\varphi_1(x)|^2+|u|^2)dx/2. \tag{36}$$

由 Cauchy-Schwarz 不等式可得

$$(3\delta-z(\theta_t\omega)-h(u))z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)uv dx+\mu z(\theta_t\omega)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)v\Delta u dx\leqslant(3\delta+|z(\theta_t\omega)|-\beta_1)|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|u||v|dx-\mu z(\theta_t\omega)\cdot$$

$\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\ \nabla u\ \nabla v\mathrm{d}x-2x\mu z(\theta_t\omega)\int_{\mathbf{R}^n}\rho'(|x|^2/r^2)\ \nabla uv\mathrm{d}x/r^2\leqslant|z(\theta_t\omega)|(3\delta-\beta_1)\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot(|u|^2+|v|^2)\mathrm{d}x/2+\sqrt{2}\mu\kappa_1|z(\theta_t\omega)|\int_{r\leqslant|x|\leqslant\sqrt{2}r}(|\nabla u|^2+|v|^2)\mathrm{d}x/r+|z(\theta_t\omega)|^2\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|u|^2+|v|^2+\mu|\nabla u|^2/2)\mathrm{d}x/2+\mu\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\nabla v|^2\mathrm{d}x.\quad(37)$

将式(31)～(37)代入式(30)，得到

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v|^2+(\delta^2-\beta_2\delta+\lambda)|u|^2+(1-\mu\delta)|\nabla u|^2+2F(x,u))\mathrm{d}x+\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot((\beta_1-2\delta)|v|^2+\delta(\lambda+\delta^2-\delta\beta_2)|u|^2+\delta(1-\mu\delta)\cdot|\nabla u|^2+\delta c_2F(x,u))\mathrm{d}x\leqslant|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)((1-\mu\delta)|\nabla u|^2\mathrm{d}x+(\lambda+\beta_1\delta-\delta^2+1/2)|u|^2+|v|^2+c_1F(x,u)/c_3)\mathrm{d}x+|z(\theta_t\omega)|(3\delta-\beta_1)\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|u|^2+|v|^2)\mathrm{d}x/2+|z(\theta_t\omega)|^2\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|u|^2+|v|^2+\mu|\nabla u|^2/2)\mathrm{d}x/2+|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|\varphi_1(x)|^2/2+c_1|\varphi_3(x)|/c_3)\mathrm{d}x+\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(\delta|\varphi_2(x)|+|g(x,t)|^2/(4\delta))\mathrm{d}x+\sqrt{2}\kappa_1\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2)/r+\sqrt{2}\kappa_1((1-\mu\delta)(\|\nabla u\|^2+\|v\|^2)+\mu(\|\nabla v^2\|+\|v\|^2))/r.\quad(38)$$

结合式(26)和式(28)可得，式(38)可化成

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v|^2+(\delta^2-\beta_2\delta+\lambda)|u|^2+(1-\mu\delta)|\nabla u|^2+2F(x,u))\mathrm{d}x\leqslant-N(t,\omega)\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v|^2+(\delta^2-\beta_2\delta+\lambda)|u|^2+(1-\mu\delta)|\nabla u|^2+2F(x,u))\mathrm{d}x+|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|\varphi_1(x)|^2/2+c_1|\varphi_3(x)|/c_3)\mathrm{d}x+\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(\delta|\varphi_2(x)|+|g(x,t)|^2/(4\delta))\mathrm{d}x+\sqrt{2}\kappa_1(\eta_3\cdot(\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2)+\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2))/r,\quad(39)$$

其中 $\eta_3=\max\{\mu,1-\mu\delta+\mu\}$.

由式(3)～(6)可知，给定一个 $\varepsilon>0$，$\exists\hat{R}_0=\hat{R}_0(\tau,\omega,\varepsilon)\geqslant1$，使得当 $r\geqslant\hat{R}_0$ 时，有

$$|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|\varphi_1(x)|^2/2+c_1\cdot|\varphi_3|/c_3)\mathrm{d}x+\delta\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\varphi_2(x)|\mathrm{d}x\leqslant\int_{|x|\geqslant r}\rho(|x|^2/r^2)(|\varphi_1(x)|^2/2+c_1|\varphi_3|/c_3)\mathrm{d}x+\delta\int_{|x|\geqslant r}\rho(|x|^2/r^2)|\varphi_2(x)|\mathrm{d}x\leqslant\varepsilon/2.\quad(40)$$

再结合式(39)和式(40)，可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v|^2+(\delta^2-\beta_2\delta+\lambda)|u|^2+(1-\mu\delta)|\nabla u|^2+2F(x,u))\mathrm{d}x\leqslant-2N(t,\omega)\cdot\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v|^2+(\delta^2-\beta_2\delta+\lambda)|u|^2+(1-\mu\delta)\cdot|\nabla u|^2+2F(x,u))\mathrm{d}x+2\sqrt{2}\kappa_1(\eta_3(\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2)+\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2))/r+\mu|z(\theta_t\omega)|\int_{\mathbf{R}^n}\rho(|x|^2/r^2)|\nabla v|^2\mathrm{d}x+\int_{\mathbf{R}^n}\rho(|x|^2/r^2)\cdot|g(x,t)|^2\mathrm{d}x/(2\delta)+\varepsilon.\quad(41)$$

在 $[\tau-t,\tau]$ 上对式(41)运用 Gronwall 不等式，得

$$\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v(\tau,\tau-t,\omega,v_0)|^2+(\delta^2-\beta_2\delta+\lambda)|u(\tau,\tau-t,\omega,u_0)|^2+(1-\mu\delta)\|\nabla u\|^2+(1-\mu\delta)|\nabla u(\tau,\tau-t,\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\omega,u_0)))\mathrm{d}x\leqslant\mathrm{e}^{2\int_\tau^{\tau-t}N(s,\omega)\mathrm{d}s}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v_0|^2+(\delta^2-\beta_2\delta+\lambda)|u_0|^2+(1-\mu\delta)|\nabla u_0|^2+2F(x,u_0))\mathrm{d}x+\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_\tau^s N(r,\omega)\mathrm{d}r}2\sqrt{2}\kappa_1(\eta_3(\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2)+\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2))\mathrm{d}s/r+\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_\tau^s N(r,\omega)\mathrm{d}r}\rho(|x|^2/r^2)|g(x,s)|^2\mathrm{d}s/(2\delta)+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_\tau^s N(r,\omega)\mathrm{d}r}\mathrm{d}s.\quad(42)$$

用 $\theta_{-\tau}\omega$ 代替 ω，则由式(42)可得

$$\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\delta^2-\beta_2\delta+\lambda)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+(1-\mu\delta)\cdot|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))\mathrm{d}x\leqslant\mathrm{e}^{2\int_\tau^{\tau-t}N(s-\tau,\omega)\mathrm{d}s}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v_0|^2+$$

$(\delta^2-\beta_2\delta+\lambda)|u_0|^2+(1-\mu\delta)|\nabla u_0|^2+2F(x,u_0))\mathrm{d}x+\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_{\tau}^{s}N(r-\tau,\omega)\mathrm{d}r}2\sqrt{2}\kappa_1(\eta_3(\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2)+\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2))\mathrm{d}s/r+\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_{\tau}^{s}N(r-\tau,\omega)\mathrm{d}r}\rho(|x|^2/r^2)|g(x,s)|^2\mathrm{d}s/(2\delta)+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{2\int_{\tau}^{s}N(r,\omega)\mathrm{d}r}\mathrm{d}s\leqslant\mathrm{e}^{2\int_0^{-t}N(s,\omega)\mathrm{d}s}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v_0|^2+(\delta^2-\beta_2\delta+\lambda)|u_0|^2+(1-\mu\delta)|\nabla u_0|^2+2F(x,u_0))\mathrm{d}x+\int_{-t}^{0}\mathrm{e}^{2\int_0^{s}N(r,\omega)\mathrm{d}r}2\sqrt{2}\kappa_1(\eta_3(\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2)+\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2))\mathrm{d}s/r+\int_{-t}^{0}\mathrm{e}^{2\int_0^{s}N(r,\omega)\mathrm{d}r}\rho(|x|^2/r^2)|g(x,s+\tau)|^2\mathrm{d}s/(2\delta)+\varepsilon\int_{-t}^{0}e^{2\int_0^{s}N(r,\omega)dr}\mathrm{d}s.$

由 $\varphi_0(\omega)\in B(\omega)$ 可知,$\exists\hat{T}_1=\hat{T}_1(\tau,\omega,D,\varepsilon)>0$,$\hat{R}_1=\hat{R}_1(\tau,\omega,\varepsilon)>1$ 使得 $\forall t>\hat{T}_1$ 和 $\forall\hat{b}\leqslant b_0$,有

$$e^{2\int_0^{-t}N(s,\omega)ds}\int_{\mathbf{R}^n}\rho(|x|^2/r^2)(|v_0|^2+(\delta^2-\beta_2\delta+\lambda)|u_0|^2+(1-\mu\delta)|\nabla u_0|^2+2F(x,u_0))\mathrm{d}x\leqslant\varepsilon. \quad (43)$$

同理可得,$\exists\hat{T}_2=\hat{T}_2(\tau,\omega,D,\varepsilon)>0$,$\hat{R}_2=\hat{R}_2(\tau,\omega,\varepsilon)>1$ 使得 $\forall t>\hat{T}_2$,$\forall r>\hat{R}_2$ 以及 $\forall\hat{b}\leqslant b_0$,由式(15)得

$$\varepsilon\int_{-\infty}^{0}\mathrm{e}^{2\int_{\tau}^{s}N(r,\omega)\mathrm{d}r}\mathrm{d}s+\int_{-\infty}^{0}\mathrm{e}^{2\int_{\tau}^{s}N(r,\omega)\mathrm{d}r}\left(\int_{|x|>r}|\rho(|x|^2/r^2)g(x,t)|^2/(2\delta)\,\mathrm{d}x\right)\mathrm{d}s\leqslant2\varepsilon. \quad (44)$$

由引理3可得,$\exists\hat{T}_3=\hat{T}_3(\tau,\omega,D,\varepsilon)>0$,$\hat{R}_3=\hat{R}_3(\tau,\omega,\varepsilon)>1$,使得 $\forall t>\hat{T}_3$,$\forall r>\hat{R}_3$,有

$$\int_{-t}^{0}\mathrm{e}^{2\int_0^{s}N(r,\omega)\mathrm{d}r}2\sqrt{2}\kappa_1(\eta_3(\|\nabla u\|^2+\|v\|^2+\|\nabla v\|^2)+\mu|z(\theta_t\omega)|(\|\nabla u\|^2+\|v\|^2))\mathrm{d}s/r\leqslant\varepsilon(1+\hat{R}_3). \quad (45)$$

令 $\hat{T}=\max\{\hat{T}_1,\hat{T}_2,\hat{T}_3\}$,$\hat{R}=\max\{\hat{R}_1,\hat{R}_2,\hat{R}_3\}$,则结合式(43)~(45)可得,$\forall t\geqslant\hat{T}$,$\forall r\geqslant\hat{R}$,有

$$\int_{|x|>\sqrt{2}r}\rho(|x|^2/r^2)(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\delta^2-\beta_2\delta+\lambda)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+(1-\mu\delta)\cdot|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))\mathrm{d}x\leqslant4\varepsilon(1+\hat{R}).$$

因此,引理5得证.

设 $\tilde{\rho}=1-\rho$,当 $r\geqslant1$ 时,令

$$\begin{cases}\tilde{u}(t,\tau,\omega,\tilde{u}_0)=\tilde{\rho}(|x|^2/r^2)u(t,\tau,\omega,u_0),\\ \tilde{v}(t,\tau,\omega,\tilde{v}_0)=\tilde{\rho}(|x|^2/r^2)v(t,\tau,\omega,v_0),\end{cases} \quad (46)$$

此时在有界域 B_{2r} 上,$\tilde{\varphi}(t,\tau,\omega,\tilde{\varphi}_0)=(\tilde{u}(t,\tau,\omega,\tilde{u}_0),\tilde{v}(t,\tau,\omega,\tilde{v}_0))$ 是方程(11)~(13)的解,其中 $\tilde{\varphi}_0=\tilde{\rho}(|x|^2/r^2)\varphi_0\in E(B_{2r})$. 式(12)乘以 $\tilde{\rho}(|x|^2/r^2)$ 且应用式(46),可得

$$\tilde{u}_t+\delta\tilde{u}-\tilde{v}=\tilde{\rho}z(\theta_t\omega),$$

$$\tilde{v}_t-\mu\Delta\tilde{v}+(h(u)-\delta)\tilde{v}-(1-\mu\delta)\Delta\tilde{u}+(\lambda+\delta^2-\delta h(u))\tilde{u}+\tilde{\rho}f(x,u)=\tilde{\rho}g(x,t)-z\tilde{v}-(z+h-3\delta)z\tilde{u}+\mu z\Delta\tilde{u}-2\mu\nabla v\nabla\tilde{\rho}-\mu v\Delta\tilde{\rho}-2(1-\mu\delta)\nabla u\cdot\nabla\tilde{\rho}-(1-\mu\delta)u\Delta\tilde{\rho}-2\mu z\nabla u\nabla\tilde{\rho}-\mu zu\Delta\tilde{\rho}. \quad (47)$$

考虑如下问题:

$$-\Delta\tilde{u}=\lambda\tilde{u},\tilde{u}|_{\partial Q_{2r}}=0, \quad (48)$$

则由方程(48)可得特征函数族$\{e_i\}_{i\in n}$,同时相应的特征值为$\{\lambda_i\}_{i\in n}$,并满足 $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_i$,$\lambda_i\to+\infty(i\to+\infty)$,由此可知$\{e_i\}_{i\in\mathbf{N}}$是 $L^2(B_{2r})$ 上的一组标准正交基.

给定 n,令 $X_n=\mathrm{span}\{e_1,e_2,\cdots,e_n\}$,$P_n:L^2(B_{2r})\to X_n$ 是投影算子,记 $Q_n=I-P_n$.

引理6 假设条件(3)~(7)和式(14)成立,则对所有的 $\varepsilon>0$,$\tau\in\mathbf{R}$,$\omega\in\Omega$ 和 $D=\{D(\tau,\omega):\tau\in\mathbf{R},\omega\in\Omega\}\in\mathscr{D}$,$\exists\tilde{T}=\tilde{T}(\tau,\omega,D,\varepsilon)>0$,$\tilde{R}=\tilde{R}(\tau,\omega,\varepsilon)\geqslant1$,$\tilde{N}=\tilde{N}(\tau,\omega,\varepsilon)>0$,使得当 $t\geqslant\tilde{T}$,$r\geqslant\tilde{R}$,$n\geqslant\tilde{N}$ 时,有

$$\|Q_n\tilde{u}(\tau,\tau-t,\theta_{-\tau}\omega)\|_{H_0^1(B_{2r})}+\|Q_n\tilde{v}(\tau,\tau-t,\theta_{-\tau}\omega)\|_{L^2(B_{2r})}\leqslant\varepsilon.$$

证 令 $\tilde{u}_{n,1}=P_n\tilde{u}$,$\tilde{u}_{n,2}=(I-P_n)\tilde{u}$,$\tilde{v}_{n,1}=P_n\tilde{v}$,$\tilde{v}_{n,2}=(I-P_n)\tilde{v}$. 对式(47)应用 $Q_n=I-P_n$,再和 $\tilde{v}_{n,2}$ 在 $L^2(B_{2r})$ 上作内积,可得

$$(1/2)\cdot\mathrm{d}\|\tilde{v}_{n,2}\|^2/\mathrm{d}t+\mu\|\nabla\tilde{v}_{n,2}\|^2+(h(u)-\delta)\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta h(u))(\tilde{u}_{n,2},\tilde{v}_{n,2})-(1-\mu\delta)(\Delta\tilde{u}_{n,2},\tilde{v}_{n,2})+(\tilde{\rho}f(x,u),\tilde{v}_{n,2})=(Q_n\tilde{\rho}g(x,t),\tilde{v}_{n,2})-z\|\tilde{v}_{n,2}\|^2-(z+h-3\delta)z(\tilde{u}_{n,2},\tilde{v}_{n,2})+\mu z(\Delta\tilde{u}_{n,2},\tilde{v}_{n,2})-2\mu(\nabla v\nabla\tilde{\rho},\tilde{v}_{n,2})-\mu(v\Delta\tilde{\rho},\tilde{v}_{n,2})-2(1-\mu\delta)\cdot(\nabla u\nabla\tilde{\rho},\tilde{v}_{n,2})-(1-\mu\delta)(u\Delta\tilde{\rho},\tilde{v}_{n,2})-2\mu z(\nabla u\nabla\tilde{\rho},\tilde{v}_{n,2})-\mu z(u\Delta\tilde{\rho},\tilde{v}_{n,2}). \quad (49)$$

对式(49)进行逐项估计可得

$(\tilde{u}_{n,2},\tilde{v}_{n,2}) \geqslant (1/2)\cdot \mathrm{d}\|\tilde{u}_{n,2}\|^2/\mathrm{d}t+\delta\|\tilde{u}_{n,2}\|^2-|z(\theta_t\omega)|\|\tilde{u}_{n,2}\|^2,\qquad(50)$

$-(\Delta\tilde{u}_{n,2},\tilde{v}_{n,2}) \geqslant (1/2)\cdot \mathrm{d}\|\nabla\tilde{u}_{n,2}\|^2/\mathrm{d}t+\delta\cdot\|\nabla\tilde{u}_{n,2}\|^2-|z(\theta_t\omega)|\|\nabla\tilde{u}_{n,2}\|^2,\qquad(51)$

$(Q_n\tilde{\rho}f(x,u),\tilde{v}_{n,2})=\mathrm{d}(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2})/\mathrm{d}t+\delta(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2})-(Q_n\tilde{\rho}f_u(x,u)u_1,\tilde{u}_{n,2})-z(\theta_t\omega)\cdot(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2}).\qquad(52)$

将式(50)～(52)代入式(49)，得

$(1/2)\cdot\mathrm{d}(\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+(1-\mu\delta)\|\nabla\tilde{u}_{n,2}\|^2+2(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2}))/\mathrm{d}t+(\beta_1-\delta)\|\tilde{v}_{n,2}\|^2+\delta(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+\delta(1-\mu\delta)\|\nabla\tilde{u}_{n,2}\|^2+\delta(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2})+\mu\|\nabla\tilde{v}_{n,2}\|^2\leqslant(\lambda+\delta^2-\delta\beta_1)|z(\theta_t\omega)|\|\tilde{u}_{n,2}\|^2+(1-\mu\delta)|z(\theta_t\omega)|\|\nabla\tilde{u}_{n,2}\|^2+|z(\theta_t\omega)|\|\tilde{v}_{n,2}\|^2+(z+\beta_2-3\delta)|z|\|\tilde{u}_{n,2}\|\|\tilde{v}_{n,2}\|+(Q_n\tilde{\rho}g(x,t),\tilde{v}_{n,2})+\mu z(\Delta\tilde{u}_{n,2},\tilde{v}_{n,2})-2\mu(\nabla v\nabla\tilde{\rho},\tilde{v}_{n,2})-\mu(v\Delta\tilde{\rho},\tilde{v}_{n,2})-2(1-\mu\delta)(\nabla u\nabla\tilde{\rho},\tilde{v}_{n,2})-(1-\mu\delta)(u\Delta\tilde{\rho},\tilde{v}_{n,2})-2\mu z(\nabla u\nabla\tilde{\rho},\tilde{v}_{n,2})-\mu z(u\Delta\tilde{\rho},\tilde{v}_{n,2})+z(\theta_t\omega)(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2})+(Q_n\tilde{\rho}f_u(x,u)u_t,\tilde{u}_{n,2}).\qquad(53)$

接下来对不等式(53)的右边进行逐项估计，由 Poincaré 不等式、Young 不等式和 Gagliardo-Nirenberg 不等式($n=3$ 的情形)得，对所有的 $\gamma<3$，有

$(Q_n\tilde{\rho}f_u(x,u),\tilde{u}_{n,2})\leqslant c\|\varphi_4(x)\|_6\|u_t\|\cdot\|\tilde{u}_{n,2}\|_3+c\|u_t\|\|u\|_6^{\gamma-1}\|\tilde{u}_{n,2}\|_{6/(4-\gamma)}\leqslant c\|u_t\|\cdot\|\tilde{u}_{n,2}\|^{1/2}\|\nabla\tilde{u}_{n,2}\|^{1/2}+c\|u_t\|\|\nabla u\|^{\gamma-1}\|\tilde{u}_{n,2}\|^{(3-\gamma)/2}\cdot\|\nabla\tilde{u}_{n,2}\|^{(\gamma-1)/2}\leqslant\delta\|\nabla\tilde{u}_{n,2}\|^2/4+c\lambda_{n+1}^{-1/2}\|u_t\|^2+c\lambda_{n+1}^{(\gamma-3)/2}\|u_t\|^2\|\nabla u\|^{2\gamma-2}\leqslant\delta\|\nabla\tilde{u}_{n,2}\|^2/4+c\lambda_{n+1}^{-1/2}(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4)+c\lambda_{n+1}^{(\gamma-3)/2}(1+\|\nabla u\|^6+\|v\|^6+|z(\theta_t\omega)|^{6/(3-\gamma)}).\qquad(54)$

由 Young 不等式可得

$(3\delta-\beta_1-z(\theta_t\omega))|z(\theta_t\omega)|\|\tilde{u}_{n,2}\|\|\tilde{v}_{n,2}\|+\mu|z(\theta_t\omega)|(\Delta\tilde{u}_{n,2},\tilde{v}_{n,2})\leqslant(3\delta-\beta_1+|z(\theta_t\omega)|)\cdot|z(\theta_t\omega)|\|\tilde{u}_{n,2}\|\|\tilde{v}_{n,2}\|+\mu|z(\theta_t\omega)|(\Delta\tilde{u}_{n,2},\tilde{v}_{n,2})\leqslant|z(\theta_t\omega)|(3\delta-\beta_1)(\|\tilde{u}_{n,2}\|^2+\|\tilde{v}_{n,2}\|^2)/2+\mu\|\nabla\tilde{v}_{n,2}\|^2+|z(\theta_t\omega)|^2(\|\tilde{u}_{n,2}\|^2+\|\tilde{v}_{n,2}\|^2+\mu\|\nabla\tilde{u}_{n,2}\|/2)/2,\qquad(55)$

$-2z\mu(\nabla u\nabla\tilde{\rho},\tilde{v}_{n,2})-\mu z(u\Delta\tilde{\rho},\tilde{v}_{n,2})\leqslant\delta\|\tilde{v}_{n,2}\|^2/2+8\sqrt{2}\mu^2|z(\theta_t\omega)|^2\kappa_1\|\nabla u\|^2/(\delta r)+8\mu^2\kappa_2|z(\theta_t\omega)|^2\cdot\|u\|^2/(\delta r^2),\qquad(56)$

$(Q_n\tilde{\rho}g(x,t),\tilde{v}_{n,2})\leqslant\|Q_n\tilde{\rho}g(x,t)\|^2/(4\delta)+\delta\|\tilde{v}_{n,2}\|^2,\qquad(57)$

$-2\mu(\nabla v\nabla\tilde{\rho},\tilde{v}_{n,2})-\mu(v\Delta\tilde{\rho},\tilde{v}_{n,2})-2(1-\mu\delta)\cdot(\nabla u\nabla\tilde{\rho},\tilde{v}_{n,2})-(1-\mu\delta)(u\Delta\tilde{\rho},\tilde{v}_{n,2})\leqslant4\sqrt{2}\kappa_1\mu(\nabla v,\tilde{v}_{n,2})/r+8\kappa_2\mu(v,\tilde{v}_{n,2})/r_2+4\sqrt{2}\kappa_1(1-\mu\delta)(\nabla u,\tilde{v}_{n,2})/r+8\kappa_2(1-\mu\delta)(u,\tilde{v}_{n,2})/r^2\leqslant\delta\|\tilde{v}_{n,2}\|^2/4+c\|\nabla v\|^2/r^2+c(\|\nabla u\|^2+(\|v\|^2+\|u\|^2)/r^2)/r^2.\qquad(58)$

结合式(53)～(58)，得到

$(1/2)\cdot\mathrm{d}(\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+(1-\mu\delta)\|\nabla\tilde{u}_{n,2}\|^2+2(Q_n\widetilde{\rho f}(x,u),\tilde{u}_{n,2}))/\mathrm{d}t+\eta_1(\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+\delta(1-\mu\delta)\|\nabla\tilde{u}_{n,2}\|^2+2(Q_n\widetilde{\rho f}(x,u),\tilde{u}_{n,2}))\leqslant|z|\eta_2\cdot(\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+(1-\mu\delta)\cdot\|\nabla\tilde{u}_{n,2}\|^2+2(Q_n\widetilde{\rho f}(x,u),\tilde{u}_{n,2}))+|z|^2\gamma_1(\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+(1-\mu\delta)\|\nabla\tilde{u}_{n,2}\|^2+2(Q_n\widetilde{\rho f}(x,u),\tilde{u}_{n,2}))/2+c\|\nabla v\|^2/r^2+c(\|\nabla u\|^2+(\|v\|^2+\|u\|^2)/r^2+8\sqrt{2}\mu^2\kappa_1|z|^2\|\nabla u\|^2/(r\delta)+8\mu^2\kappa_2|z|^2\|u\|^2/(r^2\delta))/r^2+c\lambda_{n+1}^{-1/2}(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4)+c\lambda_{n+1}^{(\gamma-3)/2}(1+\|\nabla u\|^6+\|v\|^6+|z|^{6/(3-\gamma)})+\|Q_n\tilde{\rho}g(x,t)\|^2/(4\delta)-(2|z|\eta_2+|z|^2\gamma_1)(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2}),\qquad(59)$

其中 η_1、η_2 和 γ_1 由式(26)得到.

由式(3)、Cauchy 不等式以及 Poincaré 不等式可得

$(2\eta_1-\delta+2|z|\eta_2+|z|^2\gamma_1)|(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2})|\leqslant c(1+|z|^2)\int_{\mathbf{R}^n}|u|^\gamma\tilde{u}_{n,2}\mathrm{d}x+c(1+|z|^2)\cdot\int_{\mathbf{R}^n}\varphi_1(x)\tilde{u}_{n,2}\mathrm{d}x\leqslant c(1+|z|^2)\|u\|_{2\gamma}^\gamma\|u_{n,2}\|+c(1+|z|^2)\|\varphi_1(x)\|\|\tilde{u}_{n,2}\|\leqslant c(1+|z|^2)\|\nabla u\|^\gamma\cdot\|\tilde{u}_{n,2}\|+c(1+|z|^2)\|\varphi_1(x)\|\|\tilde{u}_{n,2}\|\leqslant c(1+|z|^2)\|\nabla u\|^\gamma\lambda_{n+1}^{-1/2}\|\nabla\tilde{u}_{n,2}\|+c(1+|z|^2)\cdot\|\varphi_1(x)\|\lambda_{n+1}^{-1/2}\|\nabla\tilde{u}_{n,2}\|\leqslant\delta\|\nabla\tilde{u}_{n,2}\|^2/4+c\lambda_{n+1}^{-1}\cdot(|z|^{12/(3-\gamma)}+\|\nabla u\|^6+1).\qquad(60)$

令 $\tilde{\varphi}=\|\tilde{v}_{n,2}\|^2+(\lambda+\delta^2-\delta\beta_2)\|\tilde{u}_{n,2}\|^2+(1-\mu\delta)\|\nabla\tilde{u}_{n,2}\|^2+2(Q_n\tilde{\rho}f(x,u),\tilde{u}_{n,2})$，则结合式(59)和式(60)，可得

$$(1/2)\cdot \mathrm{d}\tilde{\varphi}(t,\omega,\tilde{\varphi}_0)/\mathrm{d}t+(\eta_1-\eta_2|z(\theta_t\omega)|-\gamma_1|z(\theta_t\omega)|^2/2)\tilde{\varphi}(t,\omega,\tilde{\varphi}_0)\leqslant 8\sqrt{2}\mu^2\kappa_1|z(\theta_t\omega)|^2\cdot\|\nabla u\|^2/(r\delta)+8\mu^2\kappa_2|z(\theta_t\omega)|^2/(r^2\delta)\|u\|^2+c\|\nabla v\|^2/r^2+c(\|\nabla u\|^2+(\|v\|^2+\|u\|^2)/r^2)/r^2+\|Q_n\tilde{\rho}g(x,t)\|^2/(4\delta)+c\lambda_{n+1}^{-1/2}(\|u\|^2+\|v\|^2+\|u\|^4+|z(\theta_t\omega)|^4)+c\lambda_{n+1}^{(\gamma-3)/2}(1+\|\nabla u\|^6+\|v\|^6+|z|^{6/(3-\gamma)})+c\lambda_{n+1}^{-1}(|z(\theta_t\omega)|^{12/(3-\gamma)}+\|\nabla u\|^6+1).$$

当 $\lambda_n\to+\infty, n\to+\infty$ 时,对于任意给定的 $\varepsilon>0$, $\exists\tilde{T}_1=\tilde{T}_1(\tau,\varepsilon,D,\omega)>0$, $\tilde{N}=\tilde{N}(\tau,\omega,\varepsilon)>0$, $\tilde{R}_1=\tilde{R}_1(\tau,\varepsilon,\omega)>1$,使得 $\forall n\geqslant\tilde{N}$ 和 $\forall r\geqslant\tilde{R}$,满足

$$(1/2)\cdot\mathrm{d}\tilde{\varphi}(t,\omega,\tilde{\varphi}_0)/\mathrm{d}t+N(t,\omega)\tilde{\varphi}(t,\omega,\tilde{\varphi}_0)\leqslant\varepsilon(1+\|\nabla u\|^6+\|v\|^6)+\varepsilon\|\nabla v\|^2+\varepsilon(1+|z(\theta_t\omega)|^{12/(3-\gamma)})+\|Q_n\tilde{\rho}g(x,t)\|^2/(4\delta).\quad(61)$$

在 $[\tau-t,\tau]$ 上对式(61)运用 Gronwall 不等式,用 $\theta_{-\tau}\omega$ 代替 ω,得到

$$\tilde{\varphi}(t,\omega,\tilde{\varphi}_0)\leqslant c\mathrm{e}^{2\int_0^{-t}N(s,\omega)\mathrm{d}s}\int_{\mathbf{R}^n}(1+\|v_0\|^2+\|\nabla u_0\|^6)\mathrm{d}x+\varepsilon\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}\|\nabla v\|^2\mathrm{d}s+\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}\|Q_n\tilde{\rho}g(s+\tau)\|^2\mathrm{d}s/(2\delta)+\varepsilon\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}(1+|z(\theta_t\omega)|^{12/(3-\gamma)})\mathrm{d}s+\varepsilon\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}(1+\|\nabla u\|^6+\|v\|^2)\mathrm{d}s.\quad(62)$$

由 $z(\theta_t\omega)$ 的缓增性可得

$$\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}(1+|z(\theta_t\omega)|^{12/(3-\gamma)})\mathrm{d}s\leqslant\int_{-t}^{0}\mathrm{e}^{\eta_1 s}(1+|z(\theta_t\omega)|^{12/(3-\gamma)})\mathrm{d}s<+\infty.\quad(63)$$

由引理3~引理5可知, $\exists\tilde{T}_2=\tilde{T}_2(\tau,\varepsilon,D,\omega)>0$, $\tilde{R}_2=\tilde{R}_2(\tau,\varepsilon,\omega)>1$,使得 $\forall t>\tilde{T}_2$, $\forall r>\tilde{R}_2$,有

$$c\mathrm{e}^{2\int_0^{-t}N(s,\omega)\mathrm{d}s}\int_{\mathbf{R}^n}(1+\|\nabla u\|^6+\|v\|^2)\mathrm{d}x\leqslant\varepsilon,\quad(64)$$

$$\varepsilon\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}(1+\|\nabla u\|^6+\|v\|^2)\mathrm{d}s\leqslant\varepsilon,\quad(65)$$

$$\varepsilon\int_{-t}^{0}\mathrm{e}^{2\int_0^sN(r,\omega)\mathrm{d}r}\|\nabla v\|^2\mathrm{d}s\leqslant\varepsilon.\quad(66)$$

由式(14)~(15)可知, $\exists\tilde{T}_3=\tilde{T}_3(\tau,\varepsilon,D,\omega)>0$, $\tilde{R}_3=\tilde{R}_3(\tau,\varepsilon,\omega)>1$,使得 $\forall t>\tilde{T}_3$, $\forall r>\tilde{R}_3$,有

$$\int_{-\infty}^{0}\mathrm{e}^{\int_0^s2N(r,\omega)\mathrm{d}r}\|Q_n\tilde{\rho}g(x,s)\|^2\mathrm{d}s\leqslant\varepsilon.\quad(67)$$

综合式(62)~(67)可得

$$\tilde{\varphi}(t,\omega,\tilde{\varphi}_0)\leqslant c.$$

令 $\tilde{T}=\max\{\tilde{T}_1,\tilde{T}_2,\tilde{T}_3\}$, $\tilde{R}=\max\{\tilde{R}_1,\tilde{R}_2,\tilde{R}_3\}$, $\forall t\geqslant\tilde{T}$, $\forall r\geqslant\tilde{R}$, $\forall n\geqslant\tilde{N}$,对于吸收集 $D\in\mathscr{D}$,且 $\varphi_0\in D(\omega)$,满足

$$\|\tilde{\varphi}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{\varphi}_{n,2,0})\|^2_{E(B_{2r})}\leqslant c\varepsilon.$$

从而引理6得证.

4 随机吸引子的存在性

最后,证明系统 Φ 存在 $\mathscr{D}$-拉回吸引子.

引理7 假设条件(3)~(7)和式(14)成立,则在 $E(\mathbf{R}^n)$ 上,方程(1)~(2)所对应的随机动力系统 Φ 是 $\mathscr{D}$-拉回渐近紧的.

证 应用引理3、引理5及引理6可证.

因此,得出本文主要结论.

定理1 假设条件(3)~(7)和式(14)成立,则系统 Φ 在 $E(\mathbf{R}^n)$ 上存在唯一的 $\mathscr{D}$-拉回吸引子.

证 由引理4可知,随机动力系统 Φ 存在 $\mathscr{D}$-拉回吸收集. 同时,由引理2和引理7可得,在 $E(\mathbf{R}^n)$ 上, Φ 是 $\mathscr{D}$-拉回渐近紧的. 则由引理1可得,在 $E(\mathbf{R}^n)$ 上 Φ 存在唯一的 $\mathscr{D}$-拉回随机吸引子.

5 参考文献

[1] Caraballo T, Langa J A, Melnik V S, et al. Pullback attractors for nonautonomous and stochastic multivalued dynamical systems [J]. Set-Valued Analysis, 2003, 11(2): 153-201.

[2] Duan Jinqiao, Schmalfuss B. The 3D quasigeostrophic fluid dynamics under random forcing on boundary [J]. Communications in Mathematical Sciences, 2003, 1(1): 133-151.

[3] Wang Bixiang. Sufficient and necessary criteria for existence of pullback attractors for noncompact random dynamical systems [J]. Journal of Differential Equations, 2012, 253(5): 1544-1583.

[4] Wang Bixiang. Random attractors for non-autonomous stochastic wave equations with multiplicative noise [J]. Discrete and Continuous Dynamical Systems, 2014, 34(1): 269-300.

[5] Wang Bixiang. Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms [J]. Stochastics and Dynamics, 2014, 14(4): 475-492.

[6] Wang Zhaojuan, Zhou Shengfan, Gu Anhui. Random attractor for a stochastic damped wave equation with multiplicative noise on unbounded domains [J]. Nonlinear

Analysis: Real World Applications，2011，12(6): 3468-3482.

[7] Crauel H，Flandoli F. Attractors for random dynamical systems [J]. Probability Theory and Related Fields，1994，100(3): 365-393.

[8] Crauel H，Flandoli F. Random attractors [J]. Journal of Dynamics and Differential Equations，1997，9(2): 307-341.

[9] Li Hongyan，You Yuncheng，Tu Junyi. Random attractors and averaging for non-autonomous stochastic wave equations with nonlinear damping [J]. Journal of Differential Equations，2015，258(1): 148-190.

[10] Lu Kening，Schmalfuss B. Invariant manifolds for stochastic wave equations [J]. Journal of Differential Equations，2007，236(2): 460-492.

[11] Wang Zhaojuan，Zhou Shengfan. Random attractor for stochastic non-autonomous damped wave equation with critical exponent [J]. Discrete and Continuous Dynamical Systems，2017，37(1): 545-573.

[12] Wang Zhaojuan，Zhang Lingping. Finite fractal dimension of random attractor for stochastic non-autonomous strongly damped wave equation [J]. Computers and Mathematics with Applications，2018，75(9): 3343-3357.

[13] Wang Bixiang. Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbf{R}^3$ [J]. Transactions of the American Mathematical Society，2008，363(7): 3639-3663.

[14] Fan Xiaoming，Wang Yaguang. Fractal dimension of attractors for a stochastic wave equation with nonlinear damping and white noise [J]. Stochastic Analysis and Applications，2007，25(2): 381-396.

[15] Yang Meihua，Duan Jinqiao，Kloeden P. Asymptotic behavior of solutions for random wave equations with nonlinear damping and white noise [J]. Nonlinear Analysis: Real World Applications，2010，12(1): 464-478.

[16] Fan Xiaoming. Random attractors for damped stochastic wave equations with multiplicative noise [J]. International Journal of Mathematics，2008，19(4): 421-437.

[17] Belleri V，Pata V. Attractors for semilinear strongly damped wave equations on $\mathbf{R}^3$ [J]. Discrete and Continuous Dynamical Systems: Series A，2012，7(4): 719-735.

The Random Attractors for Non-Autonomous Stochastic Wave Equations with Multiplicative Noise and Strongly Damped Terms on Unbounded Domains

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Abstract: The attractor for non-autonomous strongly damped stochastic wave equation with multiplicative noise on unbounded domains is studied. By using the uniform estimates and decomposition technique of the solutions for transformed system，the pullback asymptotic compactness of the corresponding system is obtained，finally getting the existence of random attractor for original system.

Key words: wave equations; random attractor; asymptotic compactness; strongly damping; unbounded domains

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