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用 q-差分算子定义的某些解析函数的性质

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摘要:利用 q-差分算子和Janowski函数定义多叶解析函数的一个新子类,该文给出类中函数的充分必要条件、系数估计、偏差定理、增长定理、凸性半径和星形性半径等几何性质.

关键词:q-差分算子;Janowski函数;多叶解析函数;偏差定理;凸性半径;星形性半径
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0　引言

q-微积分由其重要的应用而影响了一些其他学科领域的发展,许多学者在这方面上做了大量研究. F. H. Jackson[1-2]引入并研究了 q-导数和 q-积分, H. M. Srivastava[3]在几何函数理论中利用 q-微积分引入 q-超几何函数,G. Gasper等[4]研究并定义了基本超几何级数.最近,一些学者[5-16]研究了用算子定义的解析函数的性质.在上述研究的基础上,本文利用 q-差分算子和Janowski函数[17]定义多叶解析函数的一个新子类,给出类中函数的充分必要条件、系数估计、偏差定理、增长定理、凸性半径、星形性半径等几何性质.

1　预备知识

设 A_p 表示在单位圆 $D=\{z\in\mathbf{C}:|z|<1\}$ 内解析且形如

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}(p\in\mathbf{N})$$

的函数构成的函数类.特别地,当 $p=1$ 时,记 $A=A_1$.

设 $f(z)$ 和 $g(z)$ 在单位圆 D 内解析,若存在Schwarz函数 $w(z)$,使得 $g(z)=f(w(z))(z\in D)$,则称 $g(z)$ 从属于 $f(z)$,记为 $g(z)\prec f(z)$.特别地,若 $f(z)$ 是单叶解析函数,则 $g(z)\prec f(z)\Leftrightarrow g(0)=f(0),g(D)\subset f(D)$.

定义1　设函数 $p(z)$ 在 D 内解析,$p(0)=1$.若 $p(z)\prec(1+Az)/(1+Bz)(-1\leqslant B<A\leqslant 1)$,则称 $p(z)$ 在函数类 $P[A,B]$ 中.

定义2　设 $q\in(0,1)$,定义 q-数如下:

$$[\lambda]_q=\begin{cases}(1-q^{\lambda})/(1-q),\lambda\in\mathbf{C},\\ \sum_{k=0}^{\lambda-1}q^k, & \lambda\in\mathbf{N}.\end{cases}$$

特别地,当 $\lambda=0$ 时,记 $[0]_q=0$.

定义3　设 $q\in(0,1)$,$f'(0)$ 存在,定义 q-差分算子

$$\partial_q f(z)=\begin{cases}(f(qz)-f(z))/((q-1)z),z\neq 0,\\ f'(0), & z=0.\end{cases}$$

从定义3可以看出

$$\lim_{q\to 1^-}\partial_q f(z)=\lim_{q\to 1^-}(f(qz)-f(z))/((q-1)z)=f'(z);$$

当 $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ 时,

$$\partial_q f(z)=[p]_q z^{p-1}+\sum_{n=1}^{\infty}[p+n]_q a_{p+n}z^{p+n-1}(z\neq 0),$$

其中 $[p]_q=(1-q^p)/(1-q)=1+q+q^2+\cdots+q^{p-1}$.

定义4　设 $f(z)\in A_p$,$-1\leqslant B<A\leqslant 1,0\leqslant\alpha<1,q\in(0,1)$,若 $f(z)$ 满足

$(z\partial_q f(z)/([p]_q f(z))-\alpha z^2\partial_q^2 f(z)/([p]_q[p-$

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$1]_q f(z)))/(1-\alpha) \prec (1+Az)/(1+Bz)$,则称 $f(z) \in S_{p,q}(\alpha,A,B)$.

由定义 4 可得如下等价关系:

$(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z)))/(1-\alpha) \prec (1+Az)/(1+Bz) \Leftrightarrow |(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z)) - (1-\alpha))/((1-\alpha)A - B(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z))))| < 1.$ (1)

特别地,当参数 α、A、B、q、p 取特定值时,$S_{p,q}(\alpha,A,B)$ 分别为 4 个已知重要函数类:

(i) 当 $\alpha=0$ 时,$f(z) \in S^*_{p,q}[A,B]$,$S^*_{p,q}[A,B]$ 表示与 Janowski 函数相关的 q- 多叶星形函数类;

(ii) 当 $\alpha=0,A=1,B=-1$ 时,$f(z) \in S^*_{p,q}$,$S^*_{p,q}$ 表示 q- 多叶星形函数类;

(iii) 当 $\alpha=0,A=1,B=-1,q\to1^-$ 时,$f(z) \in S^*_p$,S^*_p 表示多叶星形函数类;

(iv) 当 $\alpha=0,A=1,B=-1,q\to1^-,p=1$ 时,$f(z) \in S^*$,S^* 表示单叶星形函数类.

2 主要结果

定理 1 设 $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n} \in A_p(p \geqslant 2)$,$[p-1]_q/[p]_q \leqslant \alpha < 1$,则 $f(z) \in S_{p,q}(\alpha,A,B)$ 的充分必要条件是

$\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n+1]_q - [p-1]_q) + (1-\alpha)(1+A)[p]_q[p-1]_q)|a_{p+n}| \leqslant (1-\alpha)(A-B)[p]_q[p-1]_q.$ (2)

证 假设不等式(2)成立,要使 $f(z) \in S_{p,q}(\alpha,A,B)$,只需不等式(1)成立. 由于

$|(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z)) - (1-\alpha))/((1-\alpha)A - B(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z))))| = |([p-1]_q z\partial_q f(z) - \alpha z^2\partial_q^2 f(z) - (1-\alpha)[p]_q[p-1]_q f(z))/((1-\alpha)A[p]_q[p-1]_q f(z) - B([p-1]_q z\partial_q f(z) - \alpha z^2\partial_q^2 f(z)))| = |(\sum_{n=1}^{\infty}([p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)[p]_q[p-1]_q)|a_{p+n}| \cdot z^{p+n})/((1-\alpha)(A-B)[p]_q[p-1]_q z^p - \sum_{n=1}^{\infty}(B[p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)A[p]_q \cdot [p-1]_q)|a_{p+n}|z^{p+n})| = |(\sum_{n=1}^{\infty}([p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)[p]_q[p-1]_q)|a_{p+n}| \cdot z^n)/((1-\alpha)(A-B)[p]_q[p-1]_q - \sum_{n=1}^{\infty}(B[p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)A[p]_q \cdot [p-1]_q)|a_{p+n}|z^n)| < 1,$

从而 $f(z) \in S_{p,q}(\alpha,A,B)$.

反之,若 $f(z) \in S_{p,q}(\alpha,A,B)$,由定义 4 知,$f(z)$ 满足不等式(1),即

$|(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z)) - (1-\alpha))/((1-\alpha)A - B(z\partial_q f(z)/([p]_q f(z)) - \alpha z^2\partial_q^2 f(z)/([p]_q[p-1]_q f(z))))| = |(\sum_{n=1}^{\infty}([p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)[p]_q[p-1]_q)|a_{p+n}|z^n)/((1-\alpha)(A-B)[p]_q[p-1]_q - \sum_{n=1}^{\infty}(B[p+n]_q(\alpha([p+n-1]_q - [p-1]_q) + (1-\alpha)A[p]_q[p-1]_q)|a_{p+n}|z^n)| < 1.$ (3)

因为不等式(3)对所有的 $z \in D$ 均成立,所以令 $z = \mathrm{Re}\, z \to 1^-$ 可得不等式(2).

综上所述,定理 1 得证.

由定理 1 得到如下系数估计.

推论 1 设 $[p-1]_q/[p]_q \leqslant \alpha < 1$,若 $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n} \in S_{p,q}(\alpha,A,B)$,则

$|a_{p+n}| \leqslant (1-\alpha)(A-B)[p]_q[p-1]_q/((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)(1+A)[p]_q[p-1]_q)(n=1,2,\cdots).$

结果是精确的,极值函数为

$f(z) = z^p - (1-\alpha)(A-B)[p]_q[p-1]_q z^{p+n}/((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)(1+A)[p]_q[p-1]_q).$

定理 2 设 $[p-1]_q/[p]_q \leqslant \alpha < 1$,$((1+A)[p]_q([p]_q - [p-1]_q) - ([p+1]_q)^2)/([p+1]_q)^2 < B < A \leqslant 1$,若 $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n} \in S_{p,q}(\alpha,A,B)$,则对 $|z| = r$,有

$[p]_q r^{p-1} - \tau_1 r^p \leqslant |\partial_q f(z)| \leqslant [p]_q r^{p-1} + \tau_1 r^p,$

其中 $\tau_1 = (1-\alpha)(A-B)[p]_q[p-1]_q[p+1]_q/((1+B)[p+1]_q(\alpha[p]_q - [p-1]_q) + (1-\alpha)(1+$

$A)[p]_q[p-1]_q)$. 结果是精确的,极值函数为

$$f(z)=z^p-(1-\alpha)(A-B)[p]_q[p-1]_q z^{p+1}/((1+B)[p+1]_q(\alpha[p]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q).$$

证　设 $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}$,由定义 3 知 $\partial_q f(z)=[p]_q z^{p-1}-\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|z^{p+n}$. 利用三角不等式可得

$$|\partial_q f(z)|=|[p]_q z^{p-1}-\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}\cdot z^{p+n-1}|\leqslant[p]_q|z|^{p-1}+\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|\cdot|z|^{p+n-1}.$$

因为 $|z|=r<1$,所以 $r^{p+n-1}\leqslant r^p$,故

$$|\partial_q f(z)|\leqslant[p]_q r^{p-1}+r^p\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|. \quad (4)$$

同理可得

$$|\partial_q f(z)|\geqslant[p]_q r^{p-1}-r^p\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|. \quad (5)$$

因为 $f(z)\in S_{p,q}(\alpha,A,B)$,所以,由定理 1 得

$$\sum_{n=1}^{\infty}((1+B)(\alpha[p+n-1]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q/[p+n]_q)[p+n]_q\cdot|a_{p+n}|\leqslant(1-\alpha)(A-B)[p]_q[p-1]_q.$$

从而

$$((1+B)(\alpha[p]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q/[p+1]_q)\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|\leqslant\sum_{n=1}^{\infty}((1+B)(\alpha[p+n-1]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q/[p+n]_q)[p+n]_q|a_{p+n}|,$$

因此

$$((1+B)(\alpha[p]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q/[p+1]_q)\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|\leqslant(1-\alpha)(A-B)[p]_q[p-1]_q,$$

从而

$$\sum_{n=1}^{\infty}[p+n]_q|a_{p+n}|\leqslant(1-\alpha)(A-B)[p]_q\cdot[p-1]_q[p+1]_q/((1+B)[p+1]_q(\alpha[p]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q). \quad (6)$$

将式(6) 代入式(4) 和式(5),定理 2 得证.

利用定理 2 的证明方法可以得到如下增长定理.

定理 3　设 $[p-1]_q/[p]_q\leqslant\alpha<1$,若 $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}\in S_{p,q}(\alpha,A,B)$,则对 $|z|=r$,有

$$r^p-\tau_2 r^{p+1}\leqslant|f(z)|\leqslant r^p+\tau_2 r^{p+1},$$

其中 $\tau_2=(1-\alpha)(A-B)[p]_q[p-1]_q/((1+B)\cdot[p+1]_q(\alpha[p]_q-[p-1]_q)+(1-\alpha)(1+A)\cdot[p]_q[p-1]_q)$. 结果是精确的,极值函数为

$$f(z)=z^p-(1-\alpha)(A-B)[p]_q[p-1]_q z^{p+1}/((1+B)[p+1]_q(\alpha[p]_q-[p-1]_q)+(1-\alpha)\cdot(1+A)[p]_q[p-1]_q).$$

定理 4　设 $[p-1]_q/[p]_q\leqslant\alpha<1$,$0\leqslant\sigma<p$. 若 $f(z)=z^p-\sum_{n=1}^{\infty}|a_{p+n}|z^{p+n}\in S_{p,q}(\alpha,A,B)$,则对

$$|z|<r_1=\min\{\inf_{n\geqslant1}(p(p-\sigma)((1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q)/((p+n)(p+n-\sigma)(1-\alpha)\cdot(A-B)[p]_q[p-1]_q))^{1/n},1\},$$

有 $f(z)\in C_p(\sigma)$.

证　设 $f(z)\in S_{p,q}(\alpha,A,B)$,要使 $f(z)\in C_p(\sigma)$,只需

$$(1+zf''(z)/f'(z)-\sigma)/(p-\sigma)\prec(1+z)/(1-z),0\leqslant\sigma<p,$$

上式等价于 $|(zf''(z)-(p-1)f'(z))/(zf''(z)+(1-2\sigma+p)f'(z))|<1$,化简后得

$$\sum_{n=1}^{\infty}(p+n)(n+p-\sigma)|a_{p+n}||z|^n/(p(p-\sigma))<1. \quad (7)$$

由不等式(2) 得

$$\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q)|a_{p+n}|/((1-\alpha)(A-B)[p]_q[p-1]_q)<1.$$

因此,若不等式(7) 成立,则只需

$$\sum_{n=1}^{\infty}(p+n)(n+p-\sigma)|a_{p+n}||z|^n/(p(p-\sigma))<\sum_{n=1}^{\infty}((1+B)[p+n]_q(\alpha[p+n-1]_q-[p-1]_q)+(1-\alpha)(1+A)[p]_q[p-1]_q)|a_{p+n}|/((1-\alpha)\cdot(A-B)[p]_q[p-1]_q),$$

即

$$|z|^n<p(p-\sigma)((1+B)[p+n]_q(\alpha[p+n-$$

$1]_q - [p-1]_q) + (1-\alpha)(1+A)[p]_q[p-1]_q)/((p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q\cdot[p-1]_q)$,

也就是

$$|z| < (p(p-\sigma)((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)(1+A)[p]_q[p-1]_q)/((p+n)(n+p-\sigma)(1-\alpha)(A-B)[p]_q\cdot[p-1]_q))^{1/n}.$$

定理 4 得证.

利用定理 4 的证明方法可得到在 $S_{p,q}(\alpha,A,B)$ 中函数的星形性半径.

定理 5 设 $[p-1]_q/[p]_q \leqslant \alpha < 1, 0 \leqslant \sigma < p$, 若 $f(z) = z^p - \sum_{n=1}^{\infty}|a_{p+n}|z^{p+n} \in S_{p,q}(\alpha,A,B)$, 则对

$$|z| < r_2 = \min\{\inf_{n\geqslant 1}((p-\sigma)((1+B)[p+n]_q(\alpha[p+n-1]_q - [p-1]_q) + (1-\alpha)(1+A)[p]_q[p-1]_q)/((p+n-\sigma)(1-\alpha)(A-B)[p]_q[p-1]_q))^{1/n}, 1\},$$

有 $f(z) \in S_p^*(\sigma)$.

3 结束语

q-微积分在经典的几何函数论中有着广泛应用,如通过 q-微积分研究 q-超几何函数级数. 本文利用 q-差分算子和 Janowski 函数引入多叶解析函数的一个新子类,通过给出该类子族解析函数的充分必要条件,得到了该类子族的系数估计、偏差定理、增长定理和星形半径等几何性质.

4 参考文献

[1] JACKSON F H. On q-functions and a certain difference operator [J]. Earth and Environmental Science Transactions of the Royal Society Edinburgh, 1909, 46(2): 253-281.

[2] JACKSON F H. On q-definite integrals [J]. Quarterly Journal of Pure and Applied Mathematics, 1910, 41(15): 193-203.

[3] SRIVASTAVA H M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions [C]//SRIVASTAVA H M, OWA S. Univalent Functions, Fractional Calculus and Their Applications. New York: Halsted Press, 1989: 329-354.

[4] GASPER G, RAHMAN M. Basic hypergeometric series [M]. Cambridge: Cambride University Press, 1990: 287.

[5] SEOUDY T M, AOUF M K. Convolution properties for certain classes of analytic functions defined by q-derivative operator [J]. Abstract Applied Analalysis, 2014, 2014: 846719.

[6] SEOUDY T M, AOUF M K. Coefficient estimates of new classes of q-starlike and q-convex functions of complex order [J]. Journal of Mathematical Inequalities, 2016, 10(1): 135-145.

[7] SRIVASTAVA H M, TAHIR M, KHAN B, et al. Some general classes of q-starlike functions associated with the Janowski functions [J]. Symmetry, 2019, 11(2): 292.

[8] MAHMOOD S, JABEEN M, MALIK S N, et al. Some coefficient inequalities of q-starlike functions associated with conic domain defined by q-derivative [J]. Journal of Function Spaces, 2018, 2018: 8492072.

[9] SRIVASTAVA H M, KHAN B, KHAN N, et al. Coeffcient inequalities for q-starlike functions associated with the Janowski functions [J]. Hokkaido Mathematical Journal, 2019, 48(2): 407-425.

[10] LIU Jinlin. Applications of differential subordinations for generalized Bessel functions [J]. Houston Journal of Mathematics, 2019, 45(1): 71-85.

[11] LIU Jinlin, AHUJA O P. Differential subordinations and argument inequalities [J]. Journal of the Franklin Institute, 2010, 347(8): 1430-1436.

[12] CHEN Yuru, SRIVASTAVA R, LIU Jinlin. A linear operator associated with a certain variation of the Bessel function $J_v(z)$ and related conformal mappings [J]. Journal of Pseudo-Differential Operators Applications, 2020, 11(3): 1331-1344.

[13] 邓冠铁, 王翠巧. Lipshitz 带形区域上的 Hardy 空间 [J]. 江西师范大学学报(自然科学版), 2017, 41(2): 175-179.

[14] 汤获, ZAYED H M, MOSTAFA A O, 等. 一致星象和凸象超几何函数的一些性质 [J]. 数学年刊 A 辑: 中文版, 2020, 41(1): 53-68.

[15] 龙品红, 李风军, 李星, 等. 卷入 Hohlov 算子的某解析双单叶函数子类的系数估计 [J]. 数学年刊 A 辑: 中文版, 2021, 42(4): 427-440.

[16] 陈玉如, 刘金林. 与 Bernoulli 双纽线相关的某些多叶解析函数 [J]. 扬州大学学报(自然科学版), 2020, 23(6): 18-21.

[17] JANOWSKI W. Some extremal problems for certain families of analytic functions [J]. Annales Polonici Mathematics, 1973, 28(3): 297-326.

(下转第 220 页)

The Similarity Analysis and Solutions of a Class of Population Balance Equations with Homogeneous Fragmentation Kernels

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Abstract: The similarity analysis and solutions of a class of population balance equations with homogeneous fragmentation kernels are studied in this paper. Firstly, the method of scaling transformation group is applied to a class of population balance equations with homogeneous fragmentation kernels to explore the similarity invariant variables of scaling functions. The self-similar solutions are constructed by the self-similar invariant variables. Secondly, the similarity solutions, exact explicit solutions and reduced integro-ordinary differential equations of the original equations are obtained by using group transformation of solutions and self-similar solutions. The analysis of dynamic behavior of solution is also presented. Finally, the similarity analysis results show that the scaling transformation group can not only be used for pure differential equations, but also be applied to population balance equations.

Key words: population balance equation; scaling transformation group; similarity invariant variable; similarity solution

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(上接第 202 页)

The Properties of Certain Analytic Functions Defined by q-Difference Operator

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Abstract: The new subclass of multivalent analytic functions is defined by means of q-difference operator and Janowski functions. The sufficient and necessary conditions, coefficient estimates, distortion theorems, growth theorems, radius of convexity and starlikeness of the new class are given.

Key words: q-difference operator; Janowski functions; multivalent analytic functions; distortion theorem; radius of convexity; radius of starlikeness

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